Nonstationary Signals Analysis
New opportunities from the Mathematics of Signal Processing

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INGV

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Contents

Main Goal

Given a signal $s(x)$, $x \in \mathbb{R}$, containing several oscillatory components, we want to study its time-frequency content

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- Medicine
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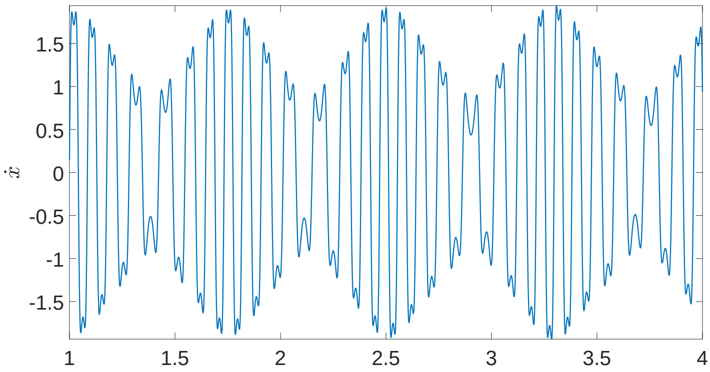
Main Problem

Real life signals are nonstationary and nonlinear: multicomponent and with features which vary over time

Ex. – Undamped Duffing Eq. – Time-Frequency Rep.



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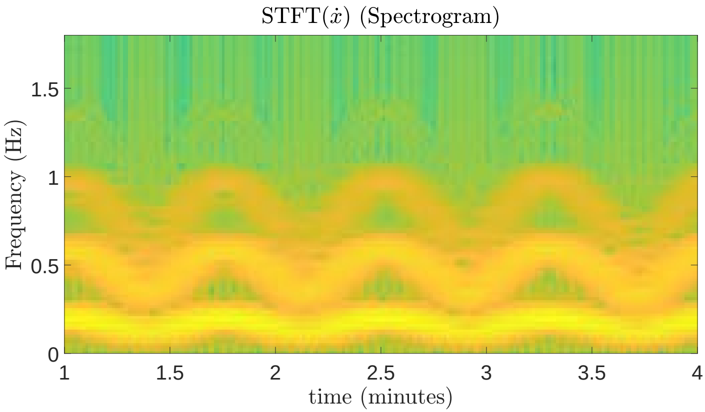

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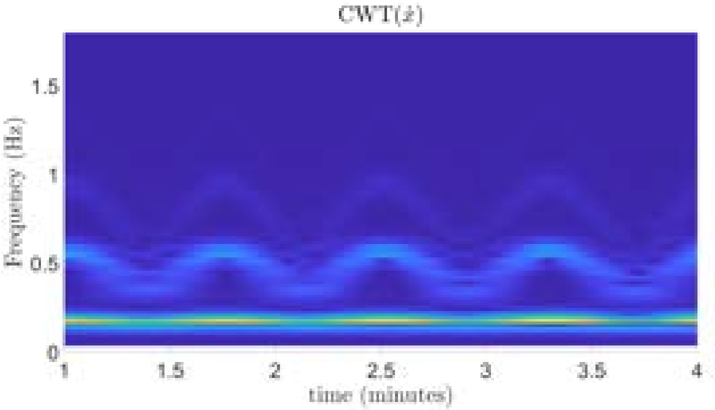

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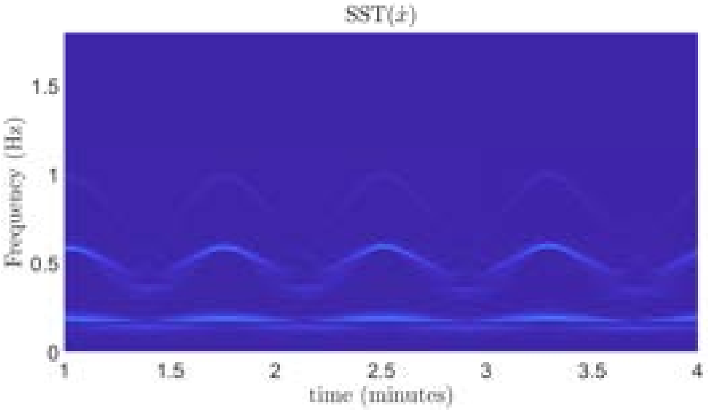

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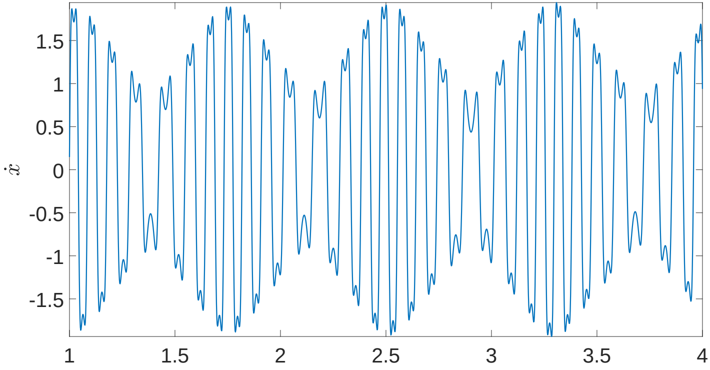

Idea for a more accurate Time-Frequency Representation

Apply a "Divide et Impera" approach
First decompose the signal into simple oscillatory components
Then study each simple oscillatory component in time-frequency separately

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What is a simple oscillatory component?

Huang's Intrinsic Mode Function – IMF

An IMF is a function s.t.

1. Number of extrema and zero crossings must either equal or differ at most by one
2. Mean value of the upper and lower envelope is zero

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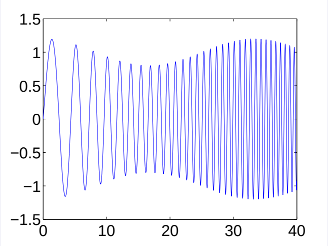

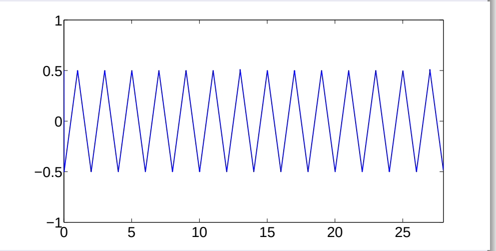

How to produce such IMFs?

- **By Optimization**
  - Sparse time–frequency representation
  - Empirical wavelet transform
  - ...

Drawback: we need to select a priori a basis

- **By Iteration**
  - Empirical Mode Decomposition – EMD
  - Ensemble EMD
  - Noise Assisted EMD
  - Iterative Filtering – IF
  - Fast Iterative Filtering – FIF

Open problems: (stability) and convergence except for IF–based methods

Contents

Hilbert-Huang Transform (HHT)

Proposed in 1998 in the paper "*The empirical mode decomposition and the Hilbert spectrum for nonlinear and non–stationary time series analysis*" by Huang and his collaborators, with **more than 16400 citations based on Scopus**. The first author, N.E. Huang, works have a total of more than 36000 citations on Scopus.

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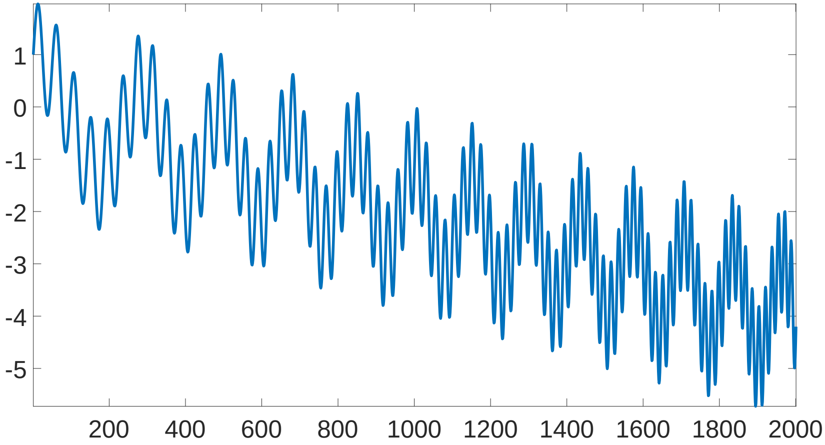
Empirical Mode Decomposition (EMD) method

1. Compute the local average m of a signal s as mean value of upper and lower envelopes
2. Subtract m from s and repeat the previous step until m becomes the zero function
3. The first IMF is given by $s - m$
4. Repeat the previous steps to produce all the IMFs

Example of decomposition via EMD



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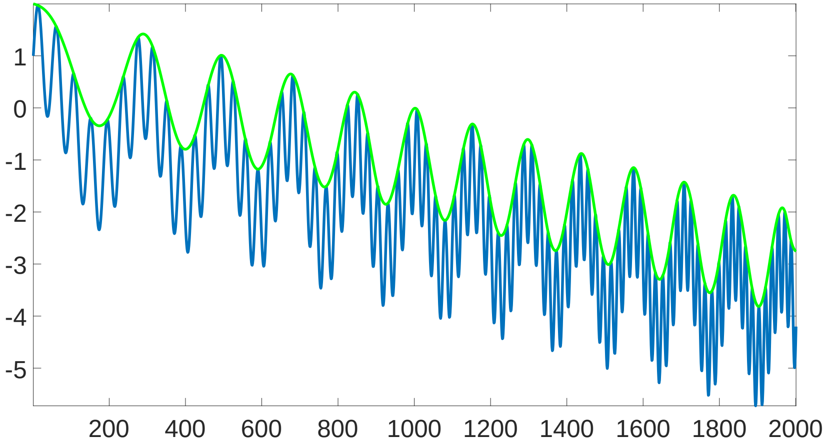

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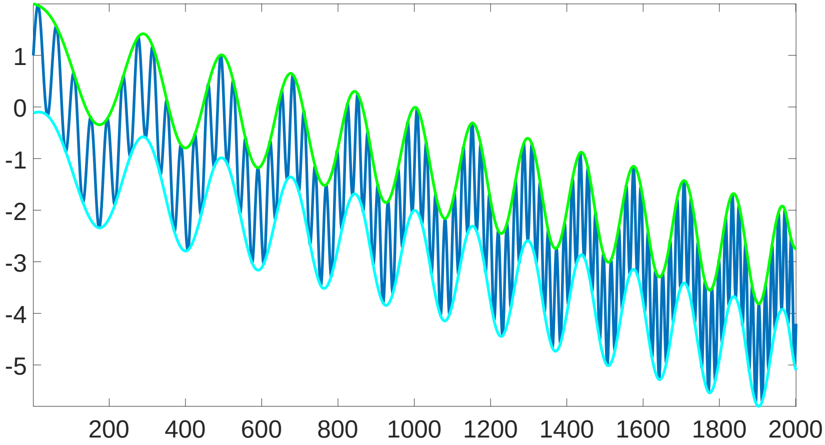

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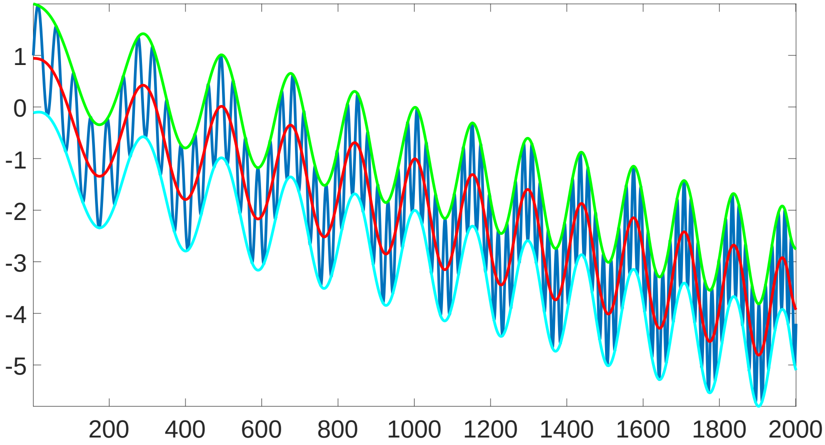

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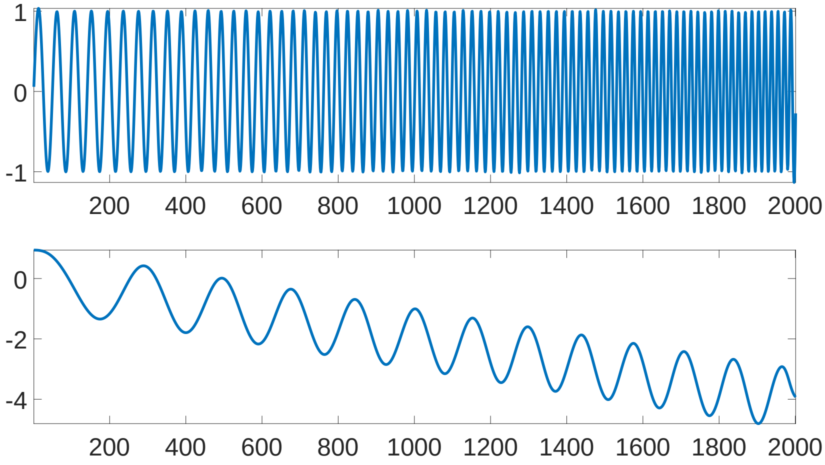

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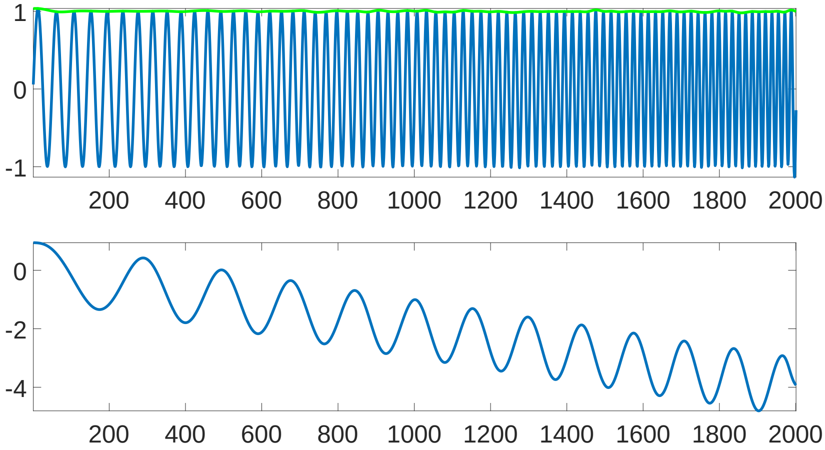

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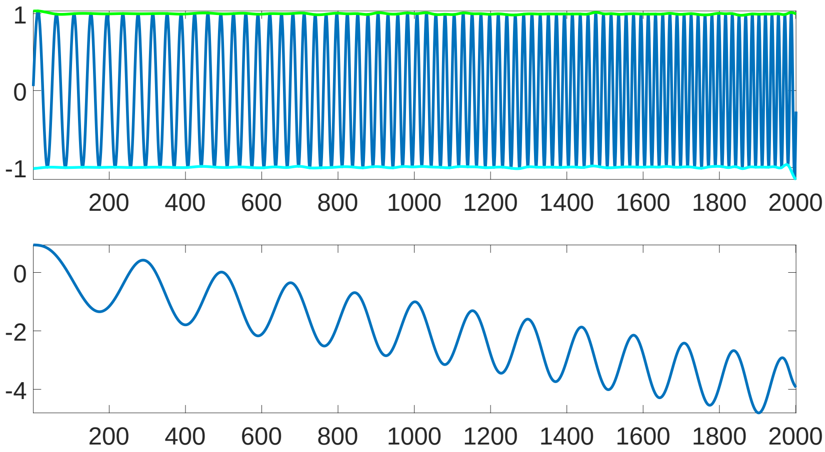

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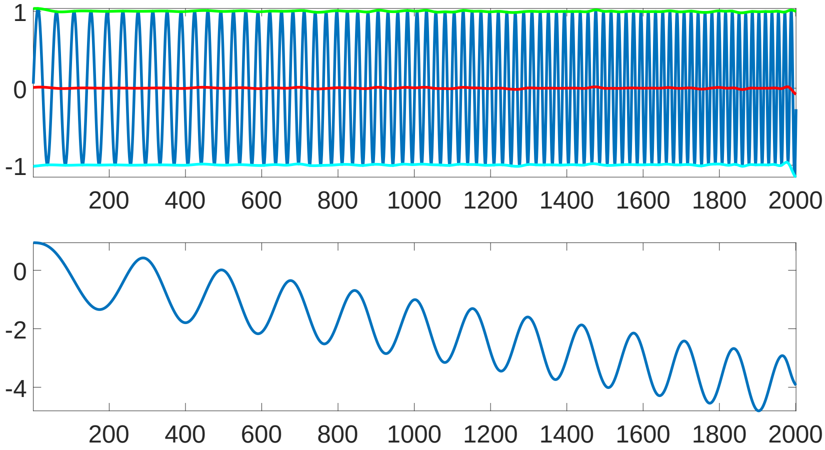

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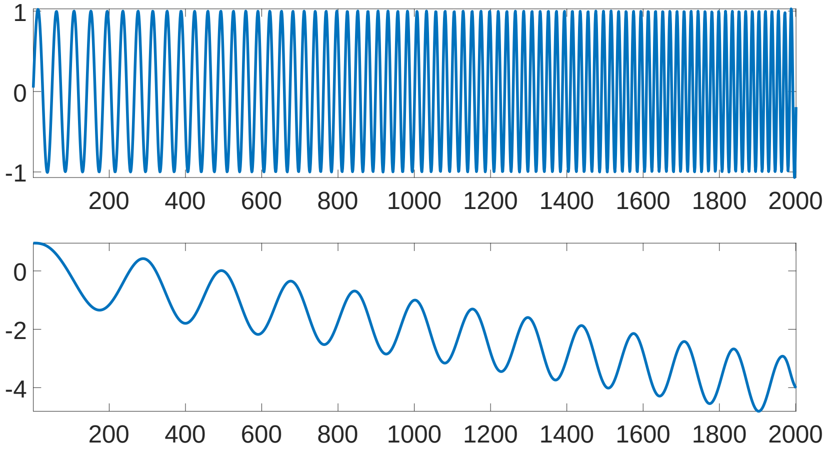

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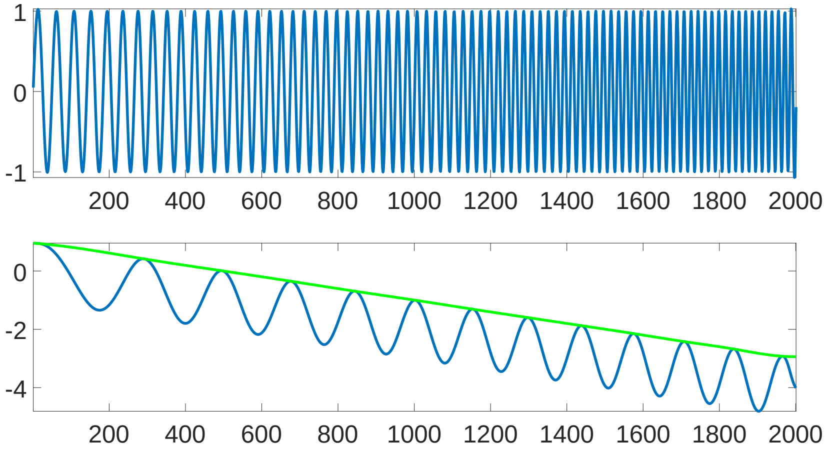

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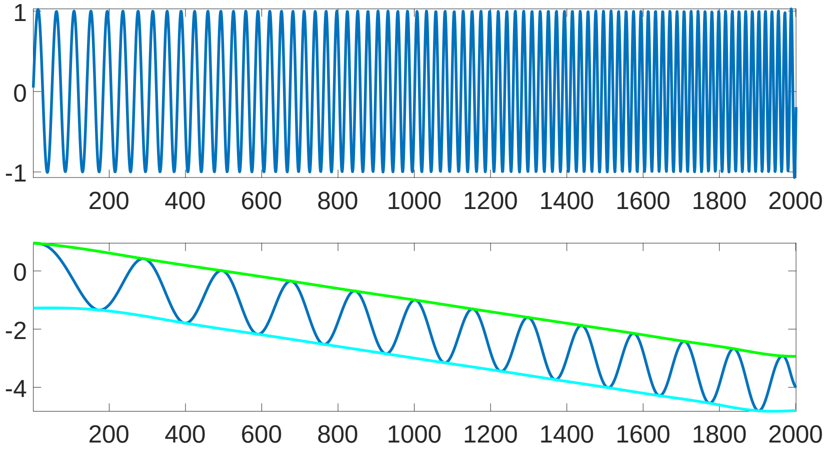

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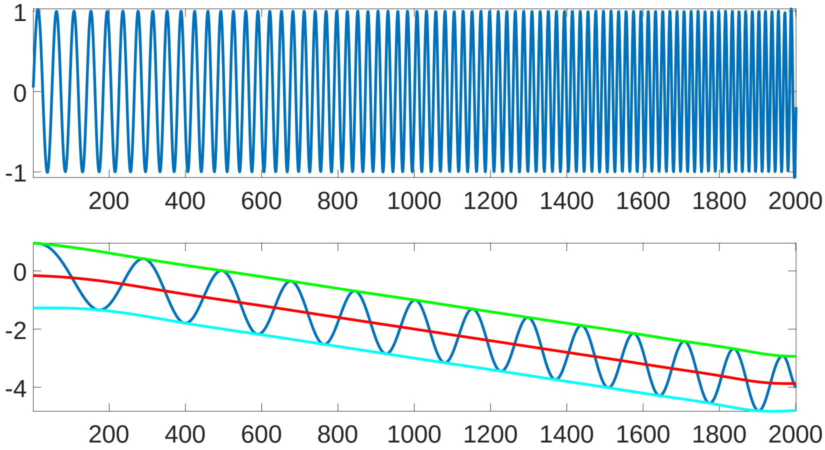

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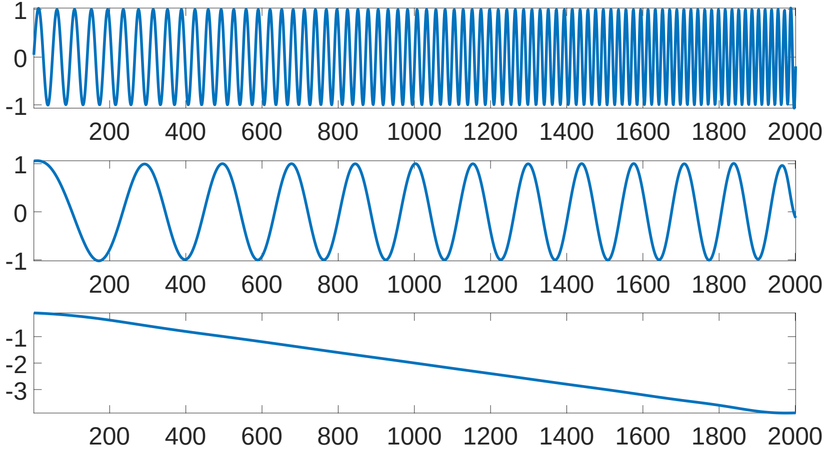

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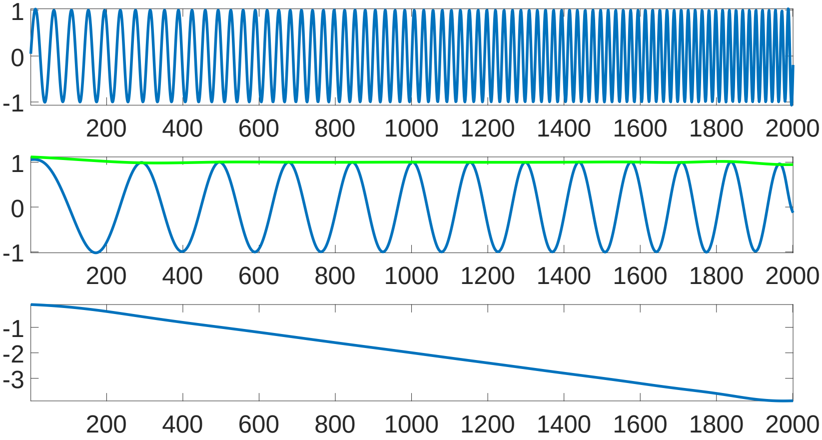

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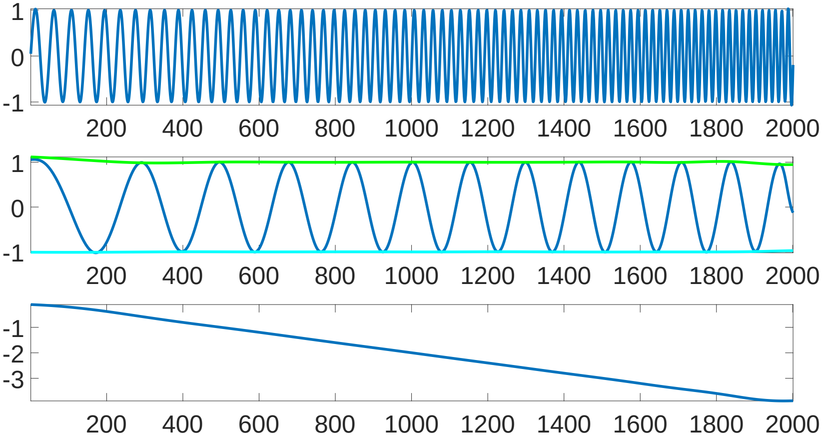

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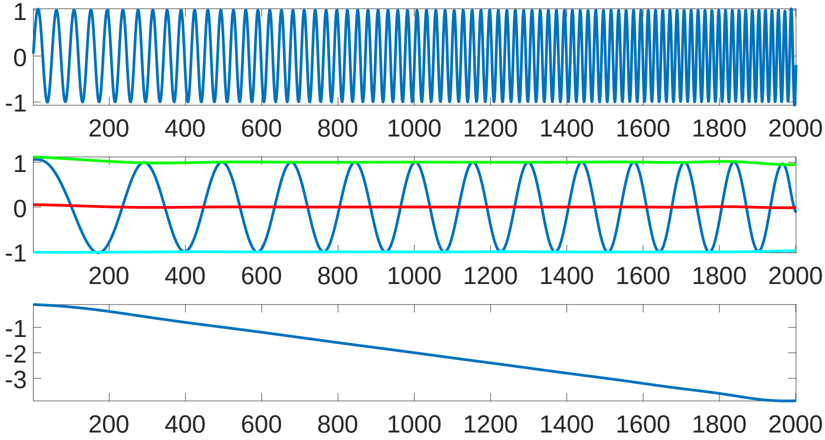

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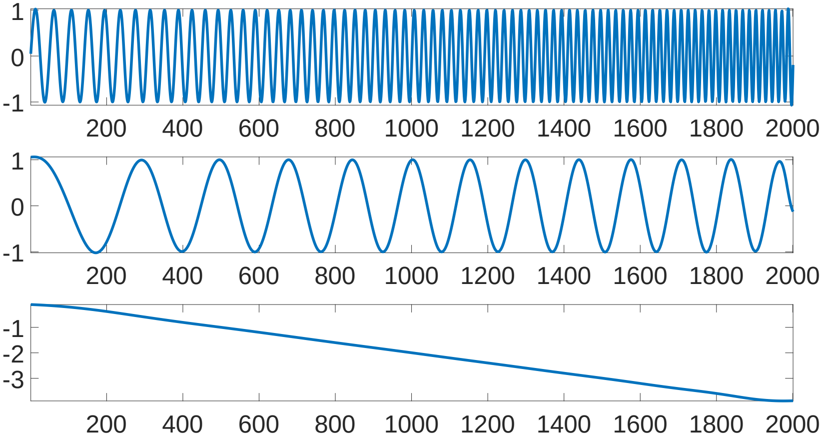

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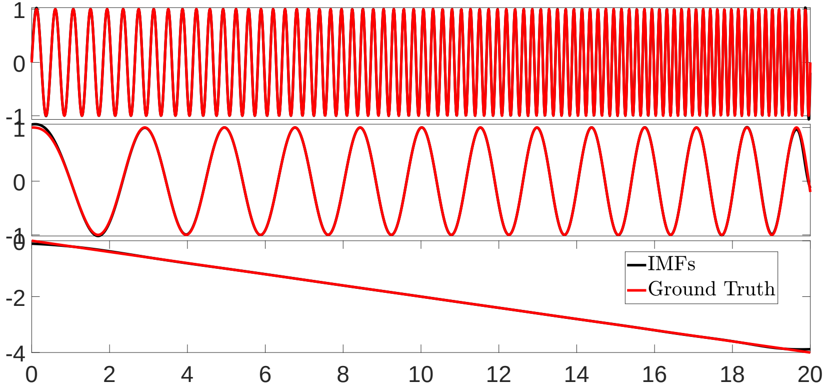

HHT - The Time-Frequency Representation

Idea

Compute each IMF instantaneous frequency via Hilbert Transform

$H(f)(x) = \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{f(\tau)}{x-\tau}\mathrm{d}\tau$ *Hilbert Transform of $f(x)$*

$z(x) = f(x) + iH(f)(x) = a(x)e^{i\theta(x)}$ *Analytic Function*

where $a(x)$, $\theta(x)$ are amplitude and phase of $z(x)$, respectively

The instantaneous frequency iF of signal $f(x)$

$$\mathrm{iF}(x) = \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}$$

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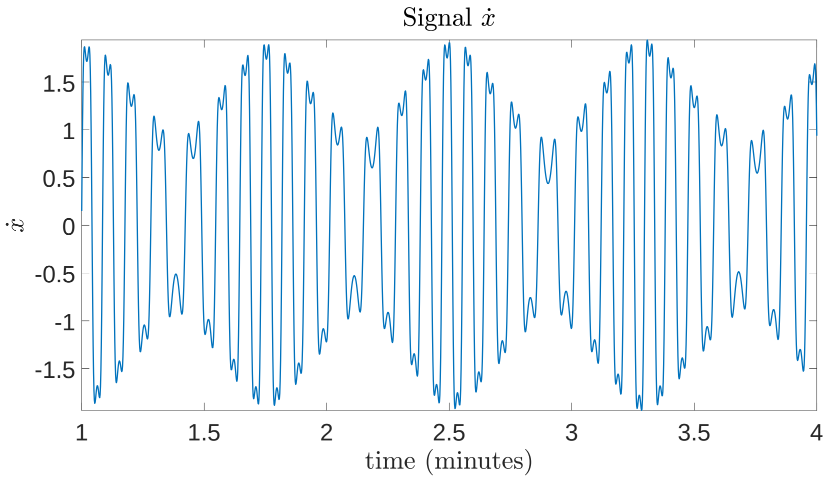

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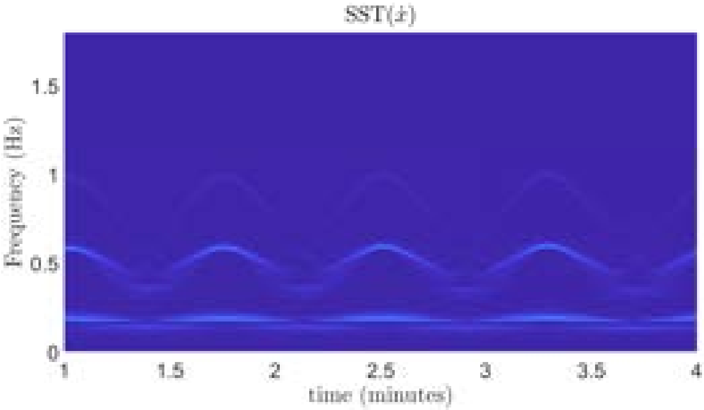

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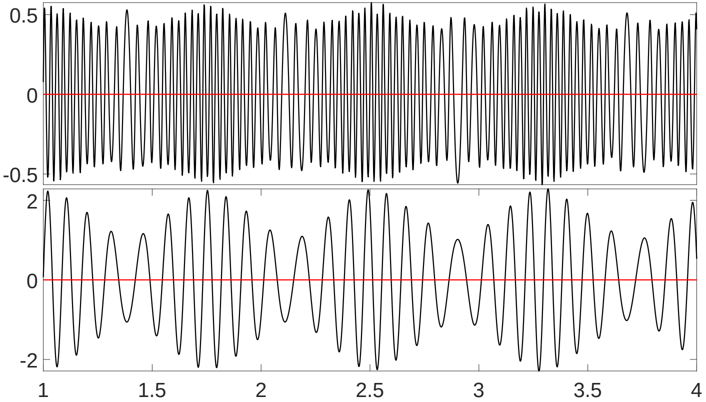

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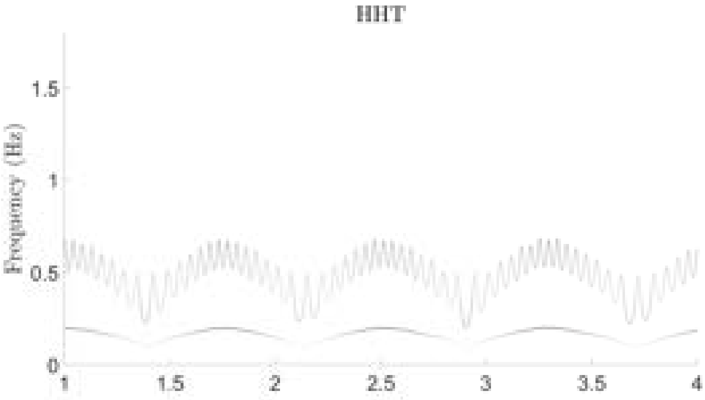

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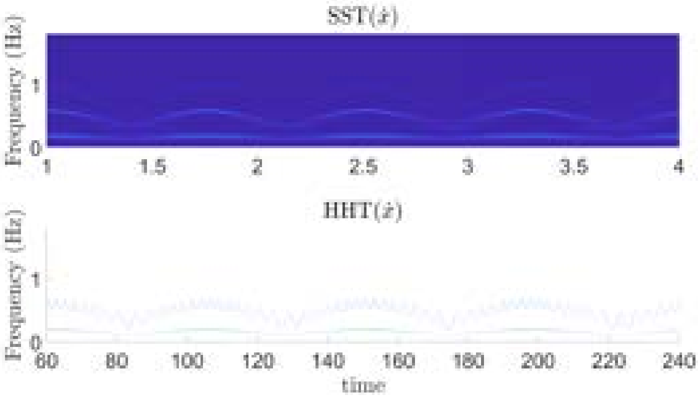

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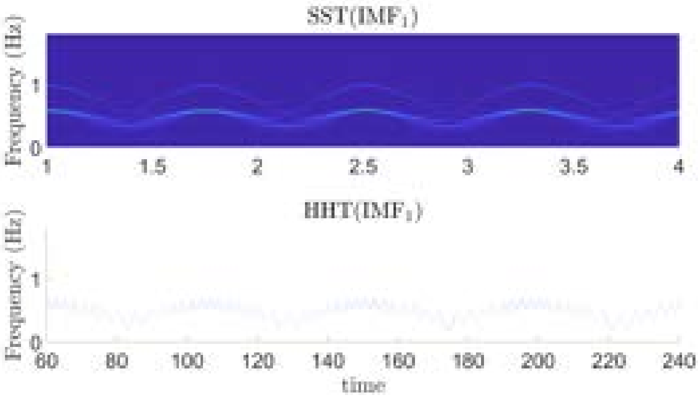

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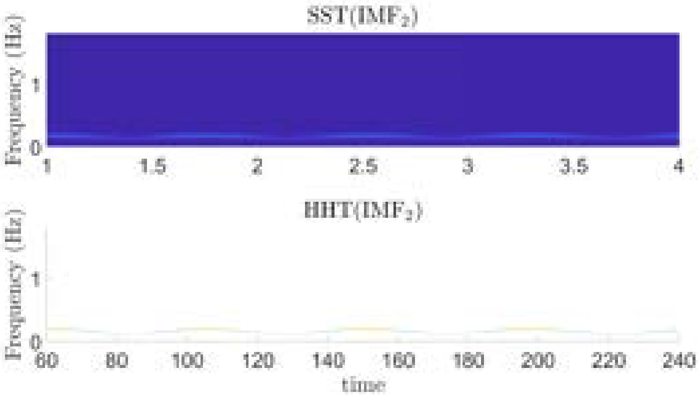

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The computation of moving averages through cubic splines that are used repeatedly in the iterations $\Rightarrow$ a small local perturbation can influence the decomposition drastically

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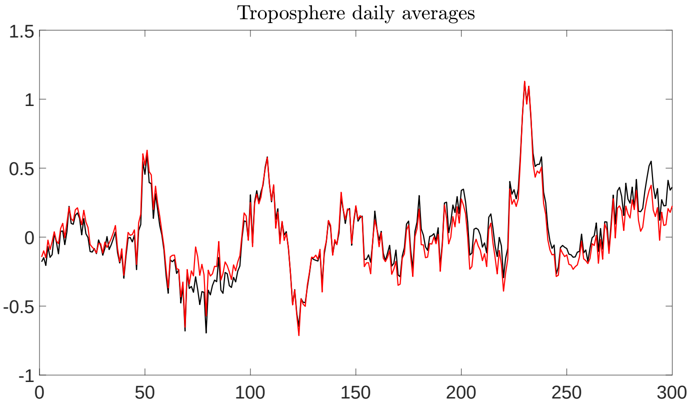

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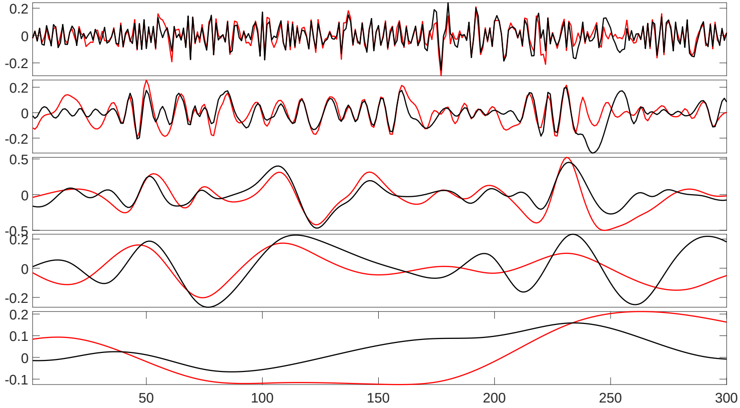

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Solved using Ensemble EMD (EEMD) and Noise Assisted EMD (NA-EMD)
Each IMF as mean of many different trials (**from 200 to 800!**)
In each trial we add a random perturbation to the original signal

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Convergence of the EMD/EEMD/NA-EMD never established

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Higher dimensions in space?

Not trivial the extension to 2D and higher dimensions in space

Contents

Iterative Filtering (IF)

Main Idea by Lin, Wang, and Zhou

Same structure of EMD algorithm with the moving average operator $\mathcal{A}$ based now on convolution

$$\mathcal{A}(s)(x) = (s * w)(x) = \int_{-L}^{L} s(x+t)w(t)\mathrm{d}t$$

where s is the signal, $w(t)$ the **filter**/**window**, and $2L$ is the support size of the filter

Nonlinear method

Given s_1 and s_1, $s_1 \neq s_2$, since $2\ell(x)$ depends on the signal itself, then in general

$$\mathrm{IMFs}(s_1 + s_2) \neq \mathrm{IMFs}(s_1) + \mathrm{IMFs}(s_2)$$

IF Convergence

Theorem – IF Convergence[1]– C., Liu, Zhou - '14

Given a window $w \in L^2([-L, L])$ and a signal $s(x) \in L^2(\mathbb{R})$.
Defined $\mathcal{H} = 1 - \mathcal{A}$, where $\mathcal{A}$ is the moving average operator.
If $|1 - \widehat{w}(\xi)| < 1$ or $\widehat{w}(\xi) = 0$, then
$\{\mathcal{H}^n(s)\}_{n\geq 1}$ converges and

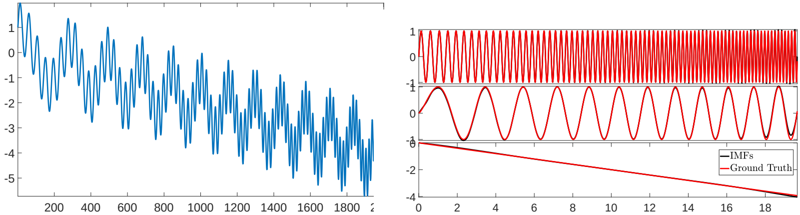
$$\lim_{n\to\infty} \mathcal{H}^n(s)(x) = \int_{-\infty}^{\infty} \widehat{s}(\xi)\chi_{\{\widehat{w}(\xi)=0\}} e^{2\pi i\xi x} \mathrm{d}\xi$$

Explicit formula for the IMF obtained using IF with filter w
We have mild sufficient conditions on the filter w that ensure the convergence of IF which are easily fulfilled

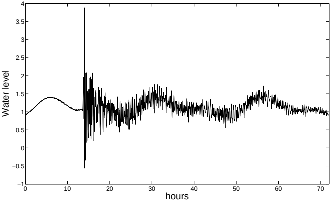
OBS: Any function given by convolution of filter with itself works

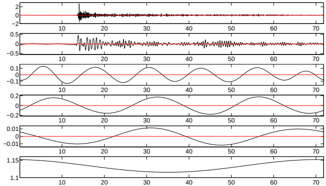
[1] Applied and Computational Harmonic Analysis - 2016

Example 1


1
0
-1
-2
-3
-4
-5
200
400
600
800
1000
1200
1400
1600
1800
IMFs
Ground Truth
0
2
4
6
8
10
12
14
16
18

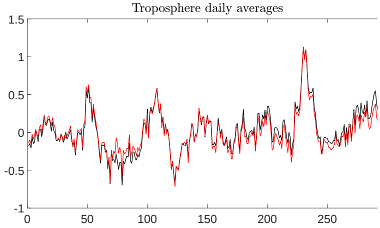
Example 2 – Tsunami water level

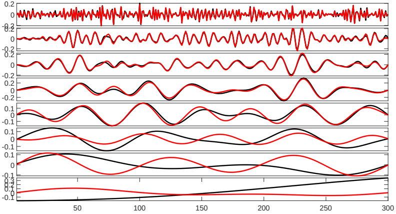




IMFs

Example 3 – Troposphere monthly mean temperature





IMFs

Fast Iterative Filtering (FIF) algorithm

Theorem – Fast Iterative Filtering (FIF) convergence[2]– C. - '20

Given $s \in \mathbb{R}^n$, a filter w and periodical extension at the boundaries, Then

$$\mathrm{IMF}_1 = U(I-D)^{N_0} U^T s = \mathrm{IDFT}\left((I - \mathrm{diag}\left(\mathrm{DFT}(w)\right))^{N_0} \mathrm{DFT}(s) \right)$$

where

D is a diagonal matrix of the eigenvalues of W

U contains the eigenvectors of the circulant matrix W

N_0 is the number of iterations needed based on a predefined stopping criterion

[2] Numerical Algorithms - 2020

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Fast calculations

The **FIF algorithm** is on average **100 times faster than IF** which is already faster than basic EMD

[2] Numerical Algorithms - 2020

Fast Iterative Filtering (FIF) Energy Conservation

L_1 Fourier Energy of a signal s

Given a signal s, its L_1 *Fourier Energy* is defined as
$E_1(s) = ||\widehat{s}||_1$, where $\widehat{s}$ is the FFT of s

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L_1 Fourier Energy conservation Theorem - C., Li, Zhou 2022

Let $s \in \mathbb{R}^n$, assuming FIF decomposes it as $s = \sum_1^m \mathrm{IMF}_k + r$, where r is a trend. Then this decomposition preserves the L_1 Fourier energy of s and produces no unwanted oscillations

Contents

The IMFogram Time–Frequency Representation

Assuming $s = \sum_{j=1}^{k} \mathrm{IMF}_j$ and ℓ_j filter length associated with IMF_j

$E_{\mathrm{IMF}_j}(t) = \frac{1}{2\eta\ell_j} \int_{t-\eta\ell_j}^{t+\eta\ell_j} \mathrm{IMF}_j(\tau)^2 \mathrm{d}\tau$, local energy of IMF_j at t

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Considering only interwave modulation frequencies, we define

The IMFogram

$$E_s(R) = \sum_{1\leq j\leq k} \frac{1}{\#\Pi_t R} \sum_{\tau\in\Pi_t R} E_{\mathrm{IMF}_j}(\tau) \mathbb{1}\left\{\Omega_f(\tau) \in \Pi_\omega R\right\}$$

R *time-frequency domain rectangular partitions*

$\Pi_t R$ are R *projection onto the time coordinate* ($\#\Pi_t R$ finite)

$\Pi_\omega R$ the *projection onto the frequency coordinate*

$\Omega_{\mathrm{IMF}_j}(t) = \frac{1}{4\eta\ell_j}$ ($\#$ IMF_j 0-crossings in $[\, t \pm \eta\ell_j \,]$) *Local Frequency*

IMFogram convergence to Spectrogram

IMFogram properties

- IMFogram contains less artifacts then alternative TFR methods

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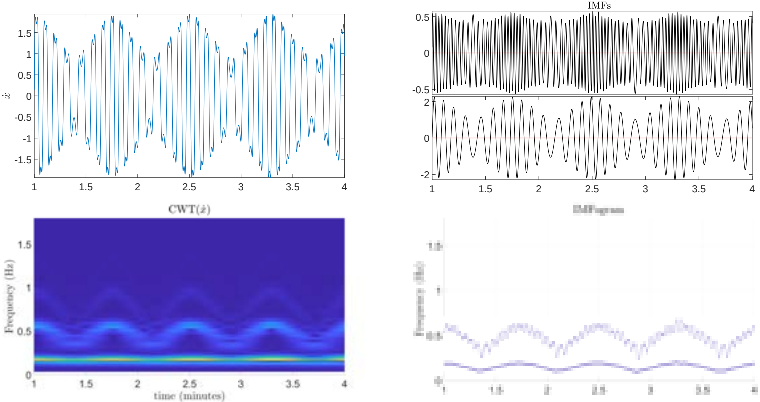
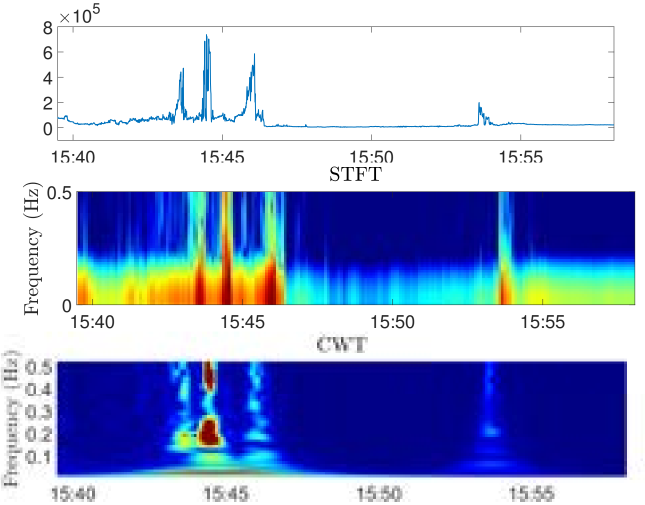
IMFogram convergence to Spectrogram Thm - C., Li, Zhou 2022

Given $s \in \mathbb{R}^p$, and K non-overlapping time windows I_i, assuming $s(I_i) = \sum_{j=1}^{N_i} a_j^{(i)} \cos(2\pi f_j^{(i)} x_k + \phi_j^{(i)})$, $x_k \in I_i$, if we let the FIF stopping criterion δ to go to zero, then the Hadamard power two of the IMFogram matrix A converges to the spectrogram matrix produced using K non-overlapping windows.

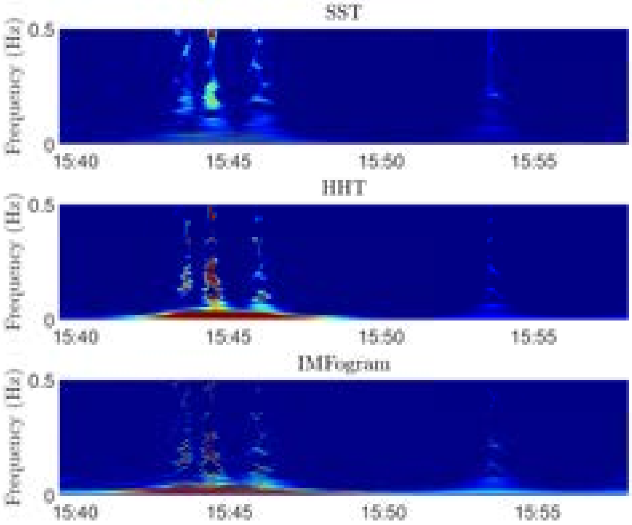
Example 1 - Undamped Duffing Equation



Example 2 - Electron density variability – ESA SWARM



Example 2 - Electron density variability – ESA SWARM



Contents

To summarize

- **FIF - Fast, stable and reliable signal decomposition**
- **L1 Fourier Energy of the signal is conserved**
- **IMFogram new time–frequency representation method**
- **Higher resolution in time and frequency**
- **Less artifacts than alternative methods**
- **Computational time comparable with Spectrogram**
- **Convergent, in the limit, to Spectrogram**
- **Window size choice appears to be not strict as in Spectrogram**
- **Fast codes (Matlab & Python) freely available online**

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  Andrea Morelli INGV, Bologna

- Two days Conference **by invitation**

Thank You for the attention

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