GW theoretical aspects and principia

Lecture 1: gravitational waves in flat spacetime

gauge freedom, effects on matter, energy flux, and source back-reaction

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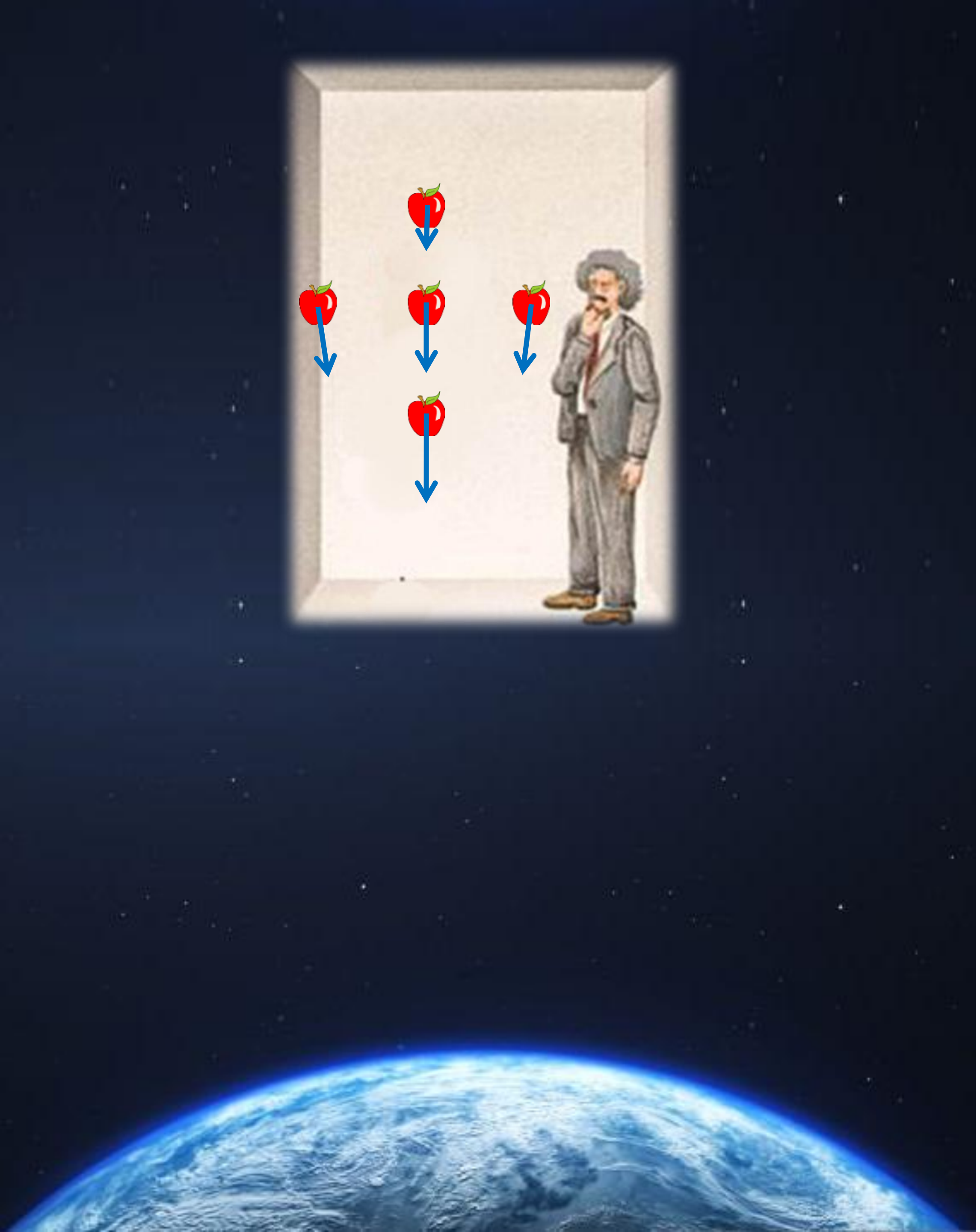
Erice, Sicily — https://indico.ego-gw.it/event/983/



The equivalence principle

- **All bodies fall in the same way in a gravitational field. This suggest the possibility of a geometric description of gravitation**
- Imagine a free-falling elevator cabin: inside it should not be possible to detect the presence of a gravitational field.
- But this is not exactly true. We expand the gravitational potential in powers of $\boldsymbol{\xi} = \boldsymbol{r} - \boldsymbol{r}_0$

$$\Phi(\mathbf{r}) = \Phi(\mathbf{r}_0) - \xi_i \mathbf{g}_i + \frac{1}{2}\xi_i \xi_j \mathcal{E}_{ij} + O(\xi^2)$$

and we obtain the equation of motion

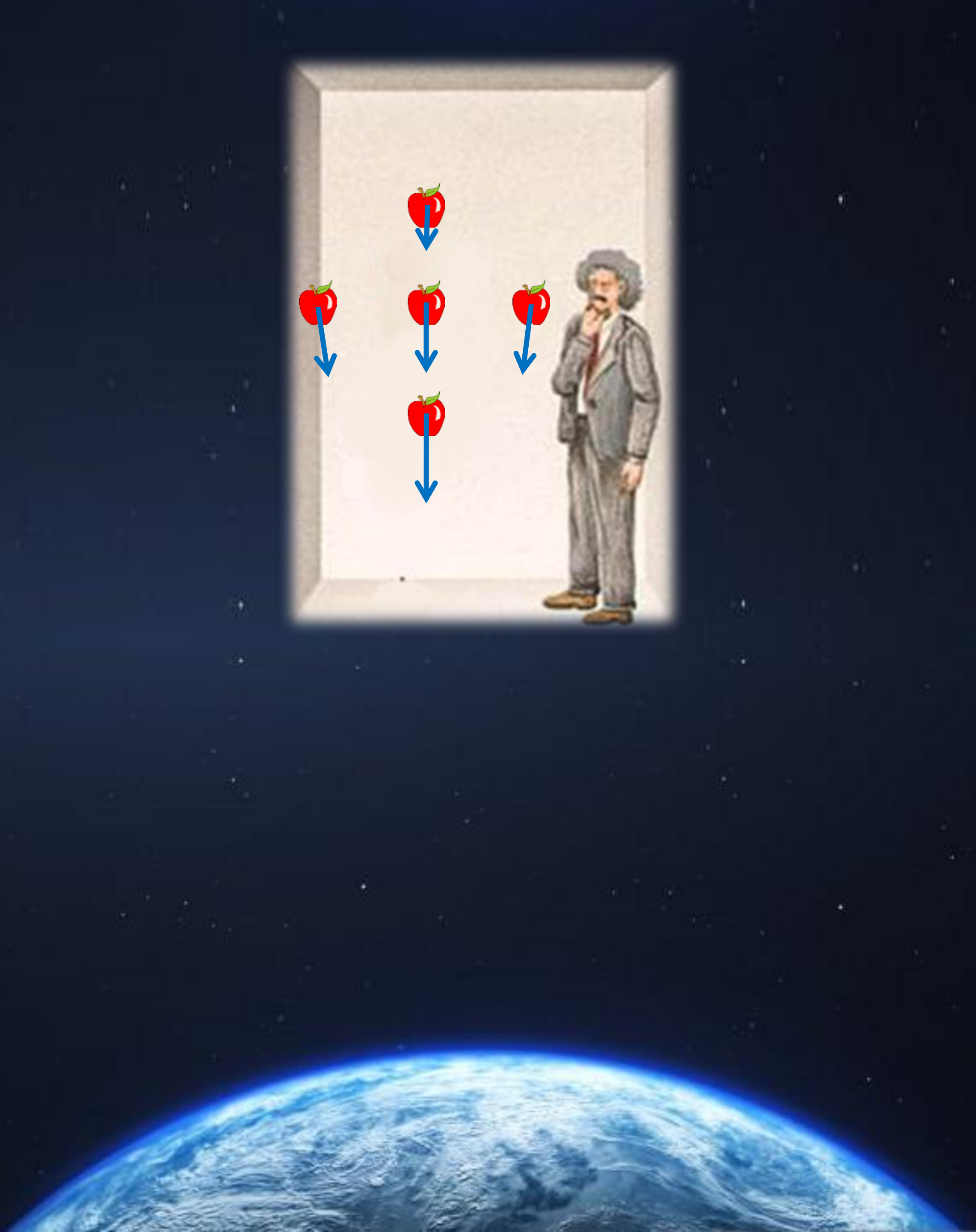
$$m\ddot{r}_i = -m\partial_i \Phi(\mathbf{r} + \boldsymbol{\xi}) = mg_i - m\mathcal{E}_{ij}\xi_j + O(\xi^2)$$

$$m\ddot{r}_{0i} = -m\partial_i \Phi(\mathbf{r}_0) = mg_i$$

i.e.

$$\ddot{\xi}_i = -\mathcal{E}_{ij}\xi_j + O(\xi^2) \qquad \mathcal{E}_{ij} = \frac{GM}{r_0^3}\left(\delta_{ij} - 3n_i n_j\right)$$

- **It is not possible to completely cancel gravitational effects with a choice of reference frame.**



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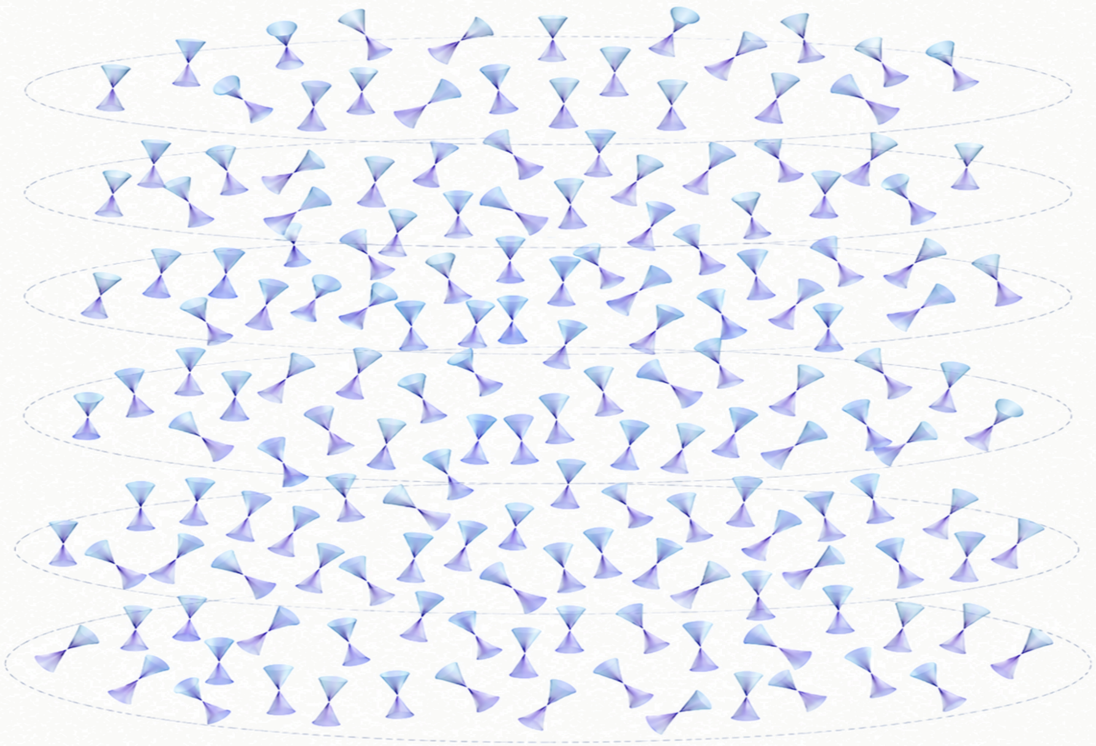
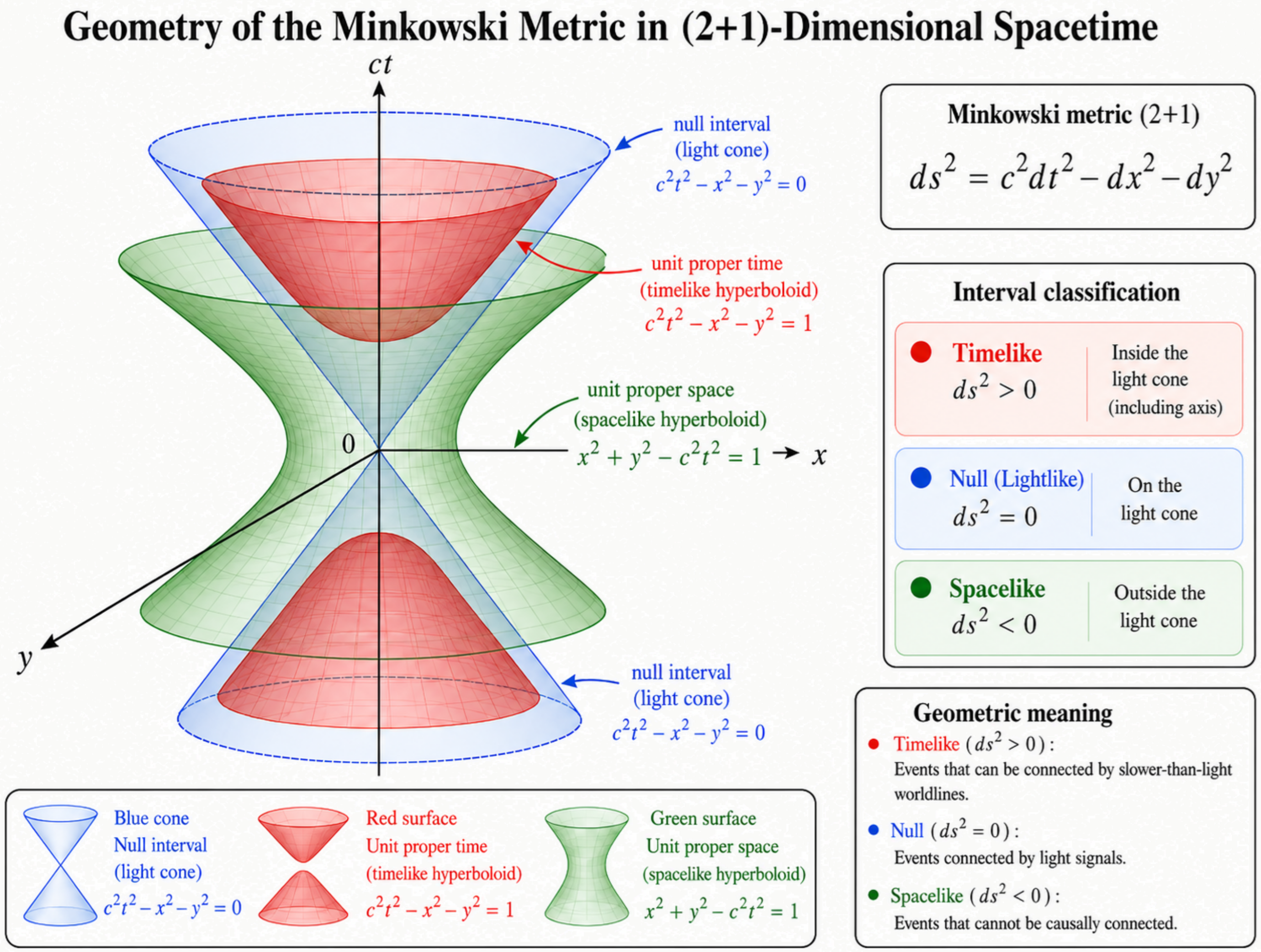
Minimal GR dictionary: metric

The geometry is described by a general metric

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

which is function of the coordinates, and determine the light cone structure of the spacetime:





Is this a non trivial geometry or a trivial geometry hidden by a bad choice of coordinates?

Minimal GR dictionary: connection

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

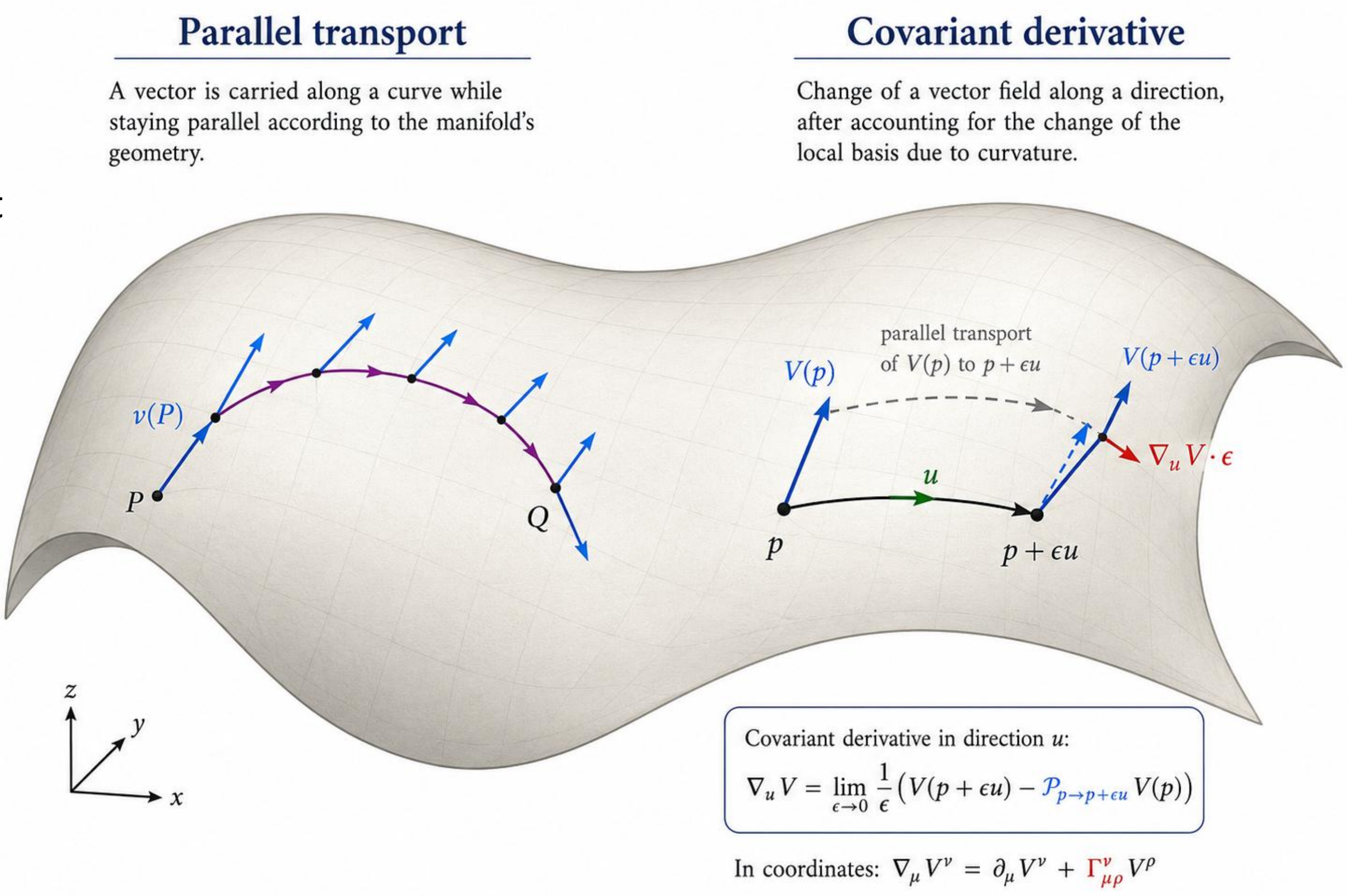
This is the Levi Civita connection, which has the unique property of preserving the scalar product of vectors under parallel transport.
It follows that $\nabla_\mu g_{\rho\sigma} = 0$

Again, $\Gamma \neq 0$ can indicate a non trivial geometry but also a "bad" choice of coordinates.

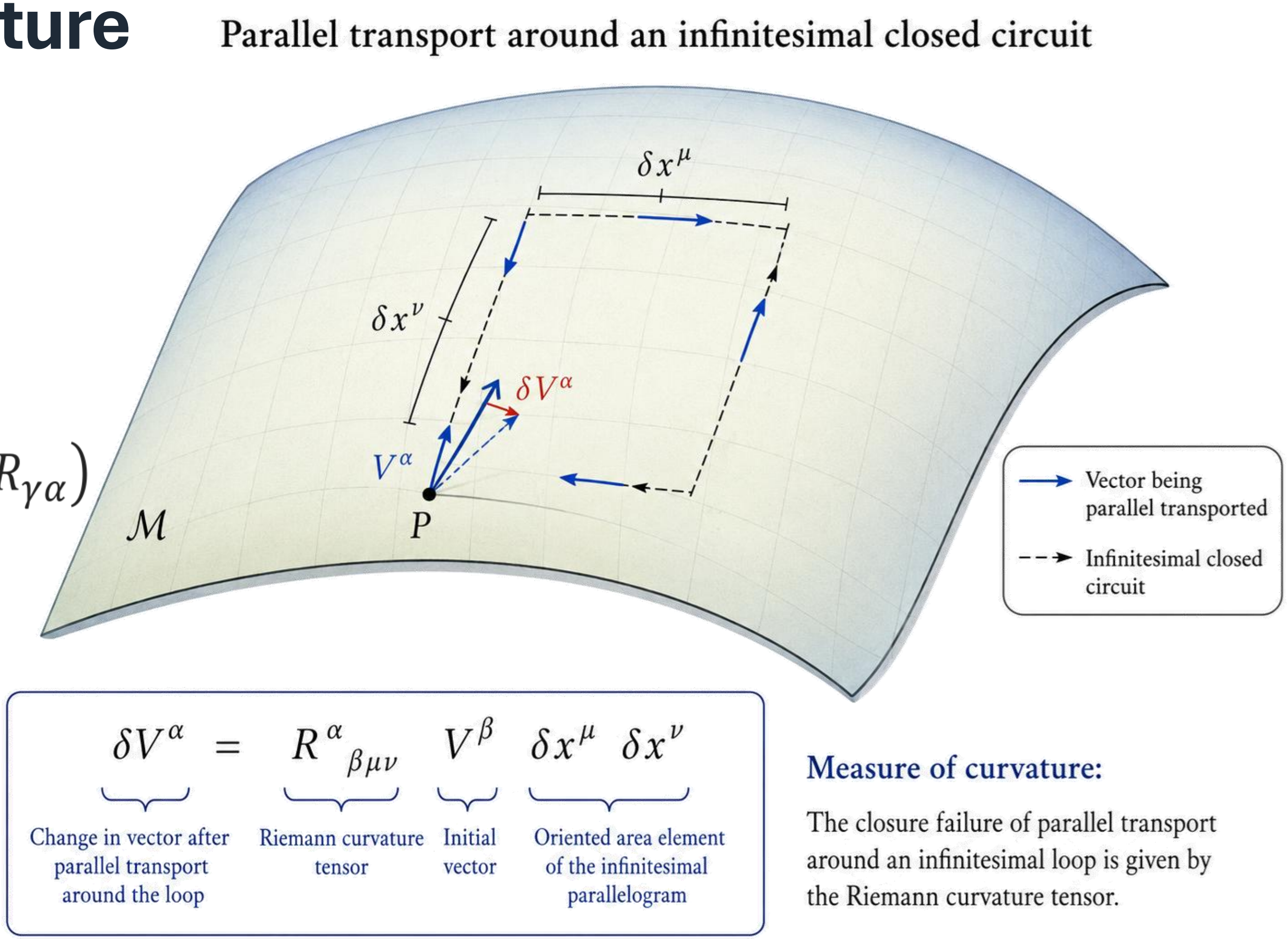
Note that covariant derivatives do not commute in general. This non commutativity is a signature of the curvature of spacetime

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

Riemann tensor



Minimal GR dictionary: curvature



Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

It can be decomposed as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= C_{\alpha\beta\gamma\delta} \\ &+ \frac{1}{2}\left(g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} - g_{\beta\gamma}R_{\delta\alpha} + g_{\beta\delta}R_{\gamma\alpha}\right) \\ &- \frac{R}{6}\left(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}\right) \end{aligned}$$

with

$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ Ricci tensor

$R = g^{\mu\nu}R_{\mu\nu}$ Scalar curvature

$C_{\mu\nu\rho\sigma}$ Weil tensor, traceless

$R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\mu\nu\sigma} + R^\rho{}_{\nu\sigma\mu} = 0$ (First Bianchi Id.)

$\nabla_\lambda R^\rho{}_{\sigma\mu\nu} + \nabla_\mu R^\rho{}_{\sigma\nu\lambda} + \nabla_\nu R^\rho{}_{\sigma\lambda\mu} = 0$ (Second Bianchi Id.)

Gravitation is geometry, not a force

In general relativity a freely falling particle follows a geodesic of the **spacetime** metric $g_{\mu\nu}$ accordingly with the equation

$$u^\nu \nabla_\nu u^\mu = 0 \qquad \text{or} \qquad \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

but it is always possible to set $\Gamma = 0$ with an appropriate choice of coordinates. So Γ does not describe the real physical content of the theory, which is encoded in the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ and in the geodesic deviation equation

$$\frac{D^2 \xi^\mu}{d\tau^2} = -R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma \qquad \text{with} \qquad \frac{D}{d\tau} = u^\nu \nabla_\nu$$

Newtonian	$\Phi(x)$	$g_i(x) = -\partial_i \Phi(x)$	$\mathcal{E}_{ij} = \partial_i \partial_j \Phi(x)$
GR	$g_{\mu\nu}(x)$	$\Gamma^\mu{}_{\nu\rho}(x)$	$R^\mu{}_{\nu\rho\sigma}$

Spacetime as a dynamical field

In GR the metric is the dynamical variable. It is governed by the Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

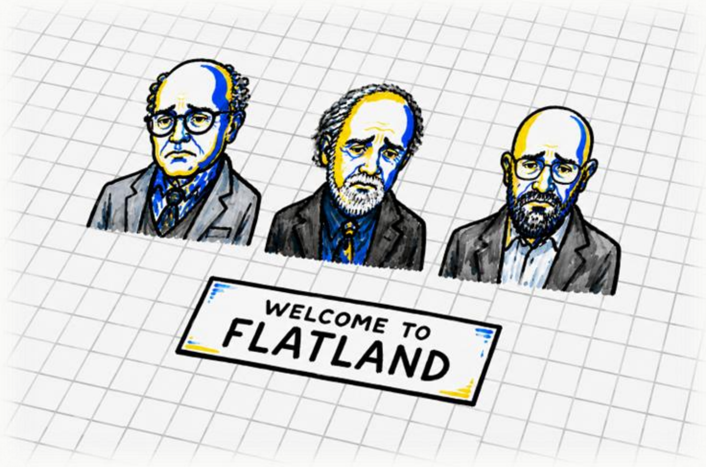
Left side is geometry, right side describe the energy, momentum and stress distribution of (non gravitational) fields.
Two important properties:

- In vacuum ($T_{\mu\nu} = 0$) the Ricci tensor and the scalar curvature are zero. Riemann tensor is equal to Weil tensor.
- As a consequence of second Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$ which imply $\nabla^\mu T_{\mu\nu} = 0$
- The second Bianchi identity imply also that

$$\nabla^\delta C_{\alpha\beta\gamma\delta} = \nabla_{[\alpha} R_{\beta]\gamma} + \frac{1}{6} g_{\gamma[\alpha} \nabla_{\beta]} R = 0$$

and becomes a propagation constraint in the vacuum.

Gravitational wave: a wave-like curvature perturbation of a spacetime.
Wave-like: has an independent dynamic, once generated. So, it can propagate in the vacuum.
Curvature: produces physical effects.
In a relativistic theory we naively expect something like this: after all, a motivation for GR is to avoid action at distance. But GW are not guaranteed to exists....



Spacetime as a dynamical field

Einstein equation gives a self consistent description of two different aspects. It connect the geometry of spacetime to its “matter” content

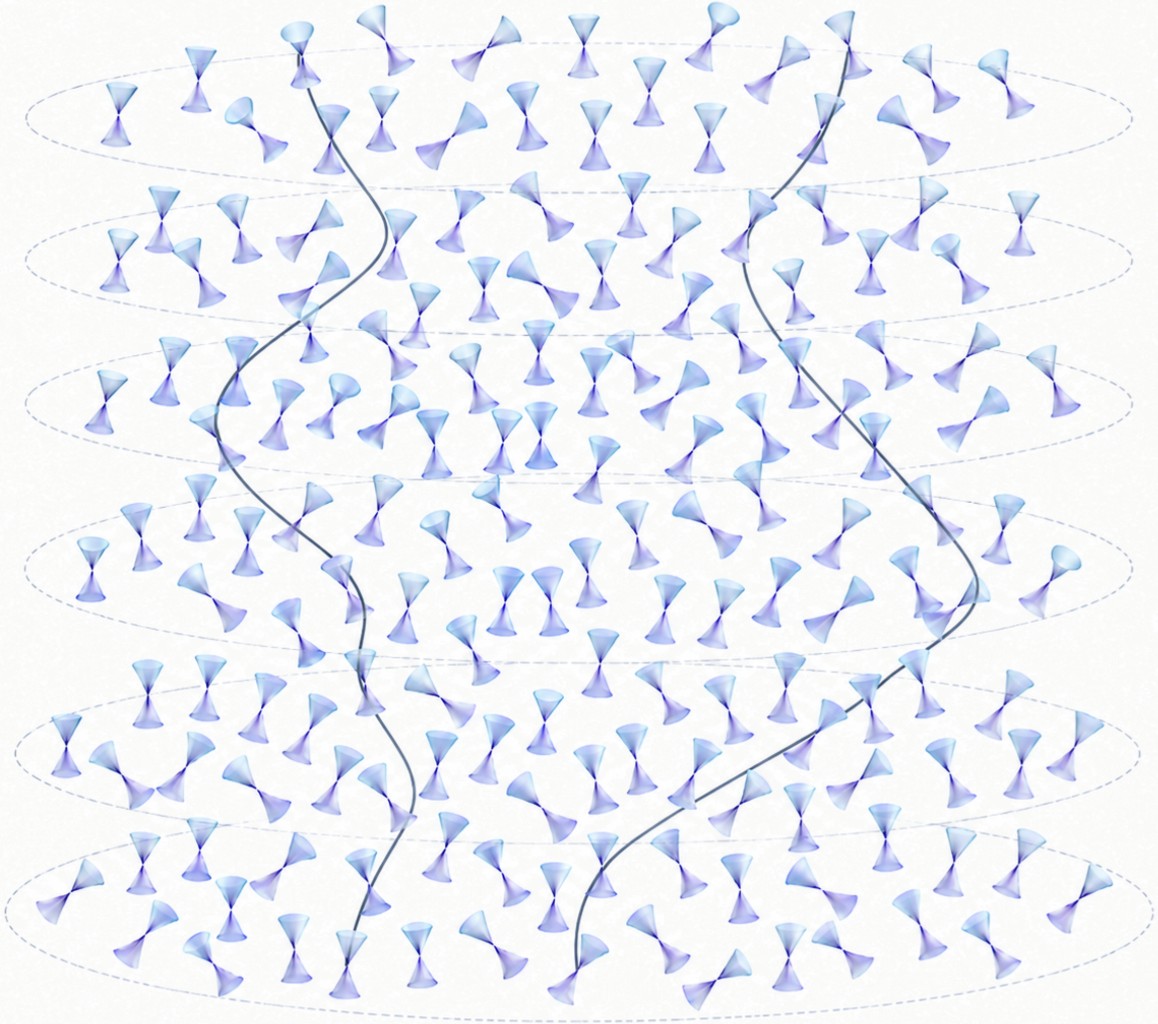
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

And determine the geodesic motion of matter in the spacetime geometry

$$\nabla^\mu T_{\mu\nu} = 0$$

Einstein equation is difficult to solve, essentially because it is nonlinear. Feasible approaches are

- Finding exact solutions with a particular symmetry
- Follow a perturbative approach around a known exact solution
- Solve the equation numerically



Geodesics and freely falling matter

A simple example: for a point particle

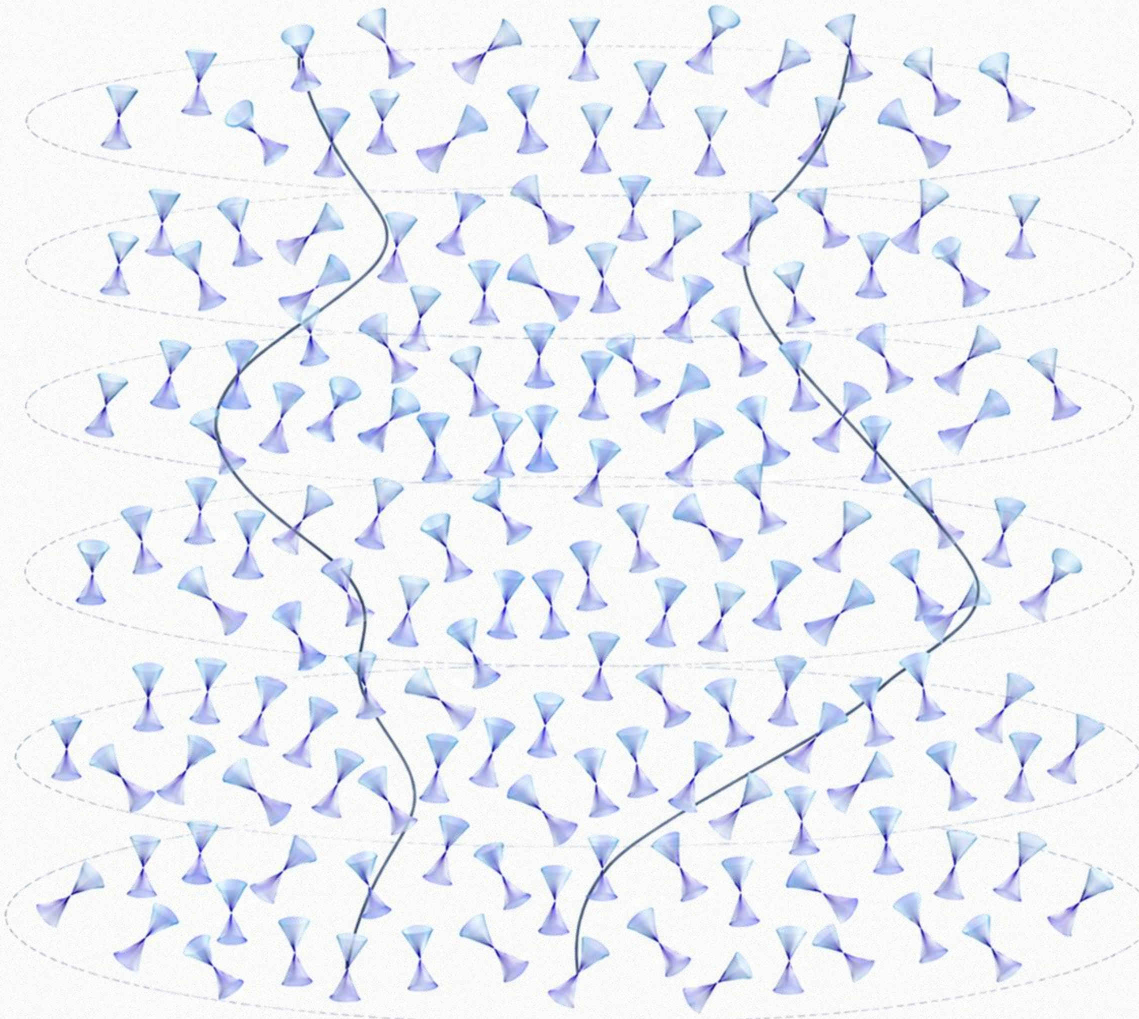
$$T^{\mu\nu}(x) = m\int u^\mu u^\nu \delta^{(4)}(x - x(\tau))d\tau$$

And determine the geodesic motion of matter in the spacetime geometry

$$0 = \langle \nabla_\mu T^{\mu\nu}, \phi_\nu \rangle = -\langle T^{\mu\nu}, \nabla_\mu \phi_\nu \rangle$$

$$= -m\int u^\mu u^\nu \nabla_\mu \phi_\nu \, d\tau = -m\int u^\nu \frac{D\phi_\nu}{d\tau} d\tau$$

$$= m\int \frac{Du^\nu}{d\tau}\phi_\nu \, d\tau \;\Rightarrow\; \frac{Du^\nu}{d\tau} = 0.$$



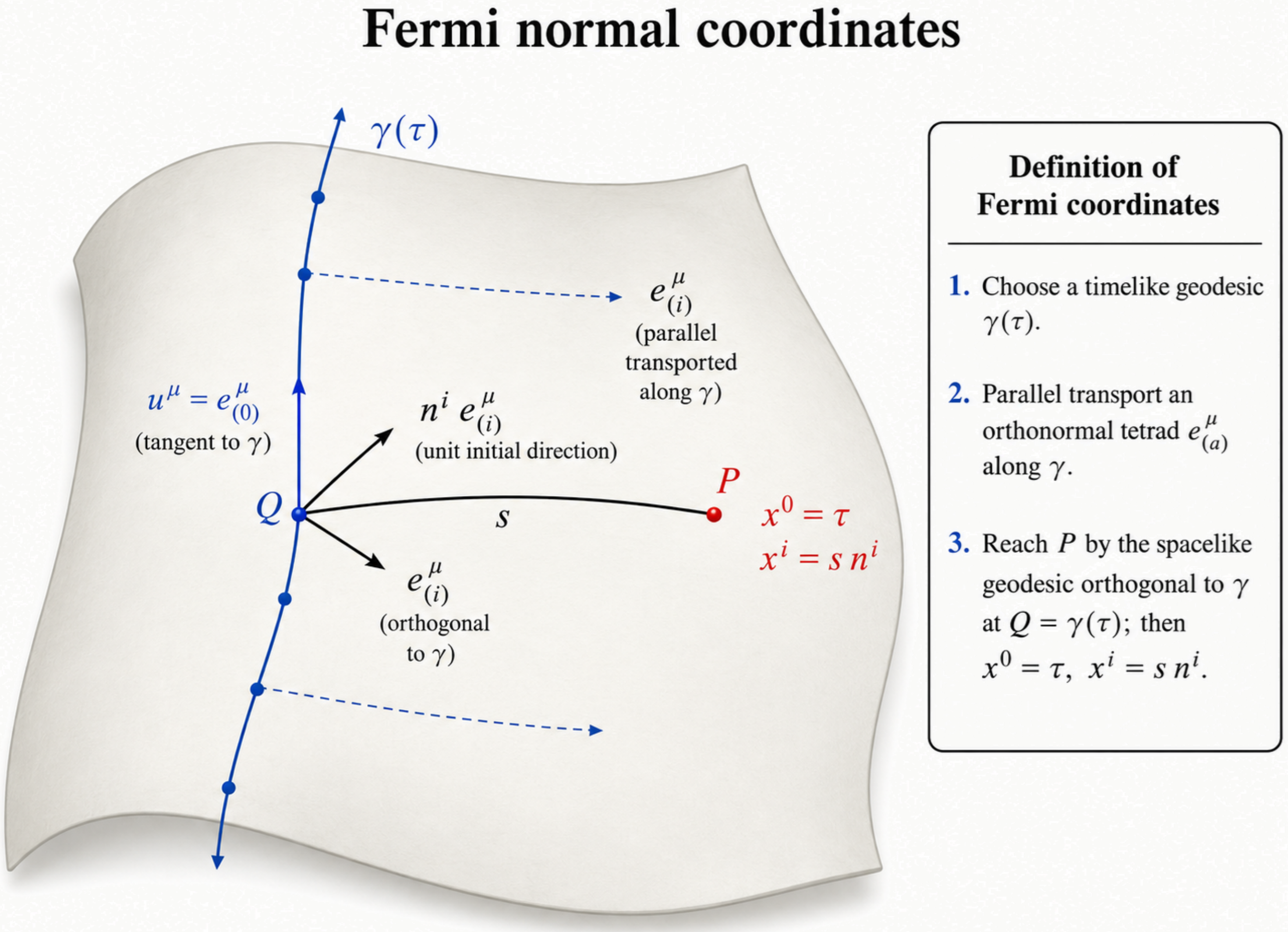
Einstein equation contains geodesic motion (note however that the metric here is given, and not perturbed by the particle)

Curvature as the observable content of gravity

$$g_{00} = -1 - R_{0i0j}(\tau)\, x^i x^j + O(x^3)$$

$$g_{0i} = -\frac{2}{3} R_{0jik}(\tau)\, x^j x^k + O(x^3)$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}(\tau)\, x^k x^l + O(x^3)$$



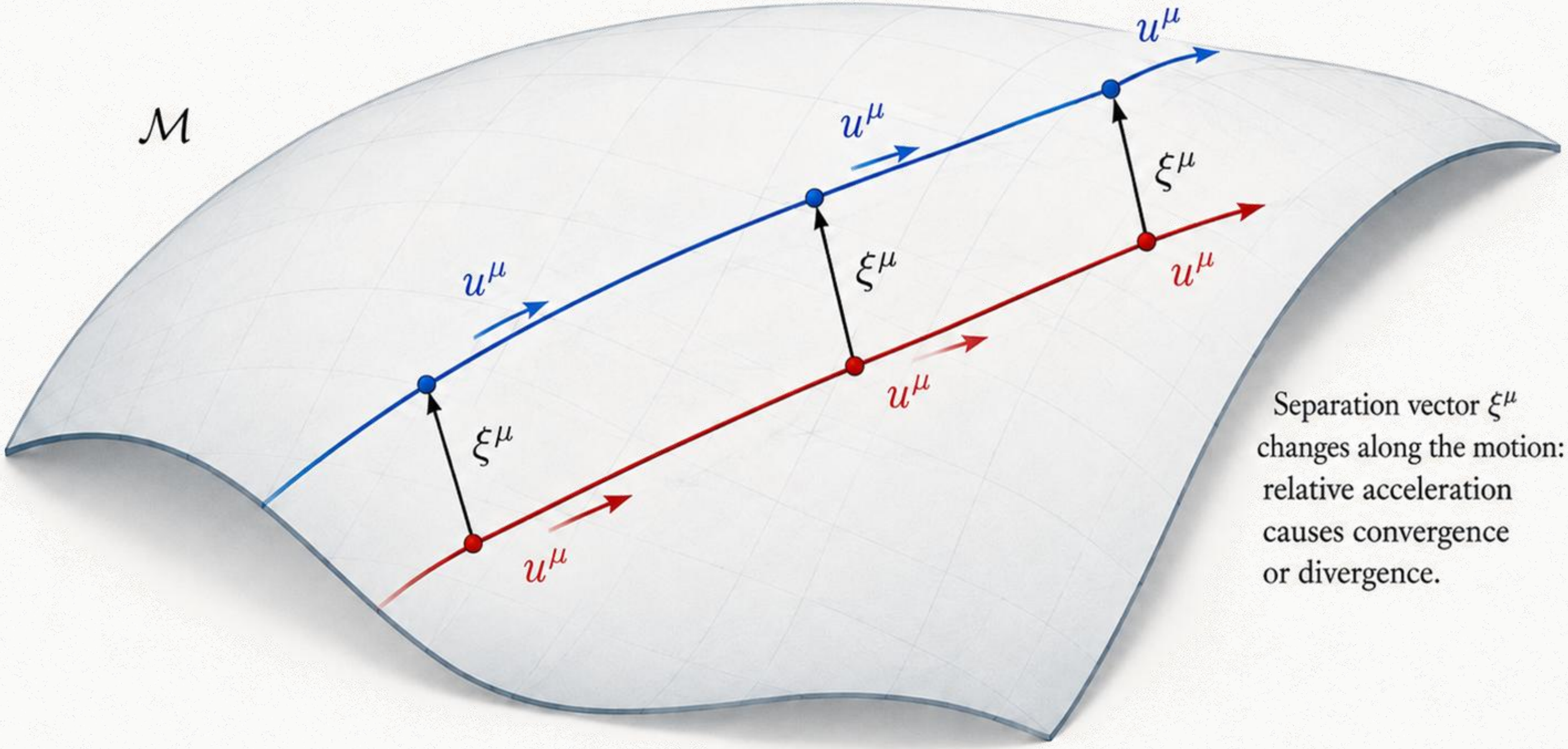
This is the metric in the natural reference frame of a free-falling observer.
$\Gamma = 0$, but **tidal effects** exist and are described by the Riemann tensor.

Geodesic deviation: how curvature acts on matter

A single freely falling particle cannot detect gravity. The measurable effect appears when we compare two (nearby) freely falling particle. Their relative acceleration is controlled by the Riemann tensor.

The equation is simple because we consider nearby particles.

Physically: it will be an accurate description in a region small compared with the typical scale of geometry



$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\alpha\beta}\, u^\nu\, \xi^\alpha\, u^\beta$$

Curvature causes nearby geodesics to converge or diverge.

Exercise: derive the geodesic deviation equation using Fermi coordinates around one of the two geodesics.

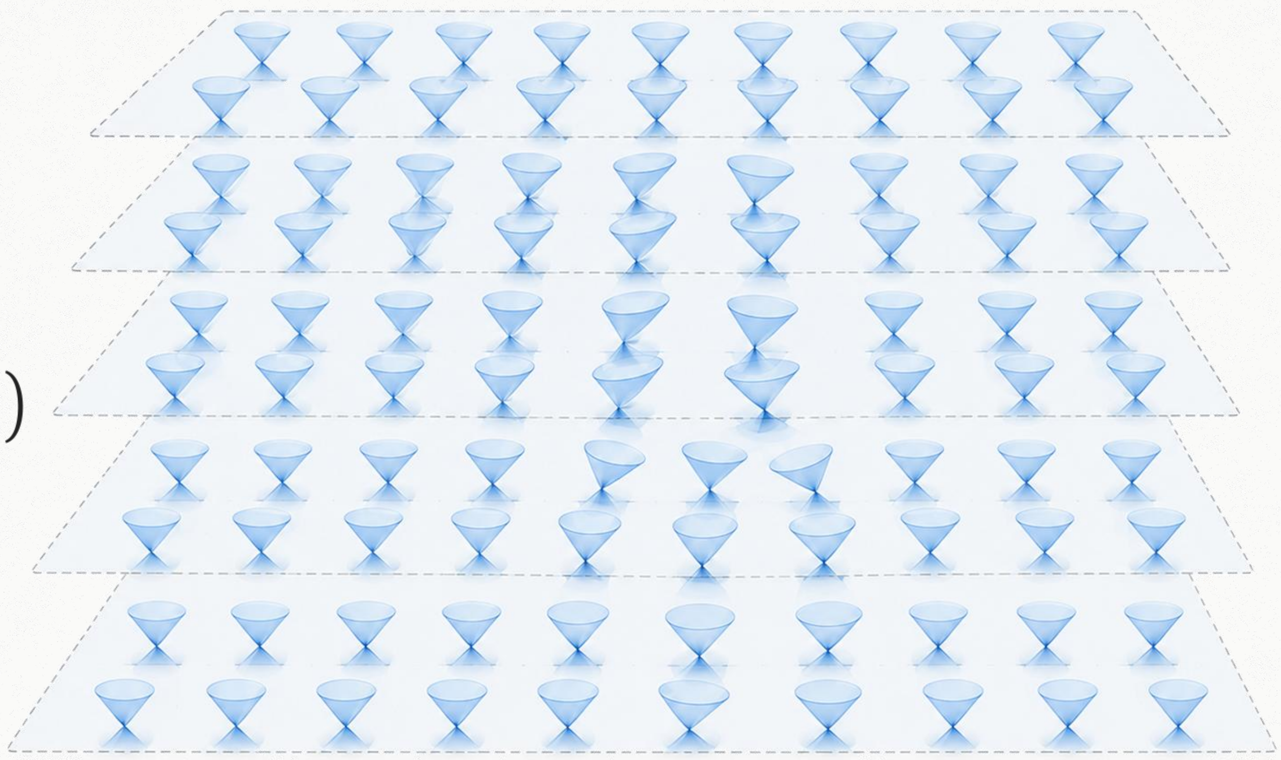
Perturbing Minkowski spacetime

Let us follow the perturbative approach and write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we suppose that quadratic terms in $h_{\mu\nu}$ are much smaller than linear ones. Note that this is not a coordinate independent statement.

If this is an acceptable approximation, then

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^{\rho}{}_{\nu} + \partial_\nu h^{\rho}{}_{\mu} - \partial^{\rho} h_{\mu\nu}\right)$$

$$R^{(1)}_{\rho\sigma\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\sigma h_{\rho\nu} + \partial_\nu\partial_\rho h_{\sigma\mu} - \partial_\mu\partial_\rho h_{\sigma\nu} - \partial_\nu\partial_\sigma h_{\rho\mu}\right)$$



Riemann tensor is a function of second derivatives of the perturbation.

Linearized Einstein equations

From the linearized Riemann tensor, we obtain the linearized Einstein tensor (here $h = \eta^{\mu\nu} h_{\mu\nu}$)

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial_\rho \partial_\mu h^\rho{}_\nu + \partial_\rho \partial_\nu h^\rho{}_\mu - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h\right)$$

And the linearized Einstein equation

$$G^{(1)}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

It is convenient to introduce the trace reversed perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - 1/2\, \eta_{\mu\nu} h$ that removes the trace terms

$$G^{(1)}_{\mu\nu} = -\frac{1}{2} \Box \bar{h}_{\mu\nu} + \underbrace{\frac{1}{2}\left(\partial_\rho \partial_\mu \bar{h}^\rho{}_\nu + \partial_\rho \partial_\nu \bar{h}^\rho{}_\mu - \eta_{\mu\nu} \partial_\rho \partial_\sigma \bar{h}^{\rho\sigma}\right)}_{\text{Divergence terms}}$$

Trace reversal and the Hilbert gauge

We are free to do "small" coordinate changes: $x^\mu \to x^{\mu\prime} = x^\mu + \xi^\mu(x)$ where $\xi^\mu = O(h)$.
Under these,

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu.$$

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial_\rho \xi^\rho$$

A divergence transforms as

$$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} - \partial_\mu \partial^\mu \xi_\nu - \partial^\mu \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\mu \partial_\rho \xi^\rho = \partial^\mu \bar{h}_{\mu\nu} - \Box \xi_\nu$$

If we can find a solution of

$$\Box \xi_\nu = \partial^\mu \bar{h}_{\mu\nu}$$

we can impose the Lorenz (also, de Donder, harmonic, Hilbert) condition, removing divergence terms

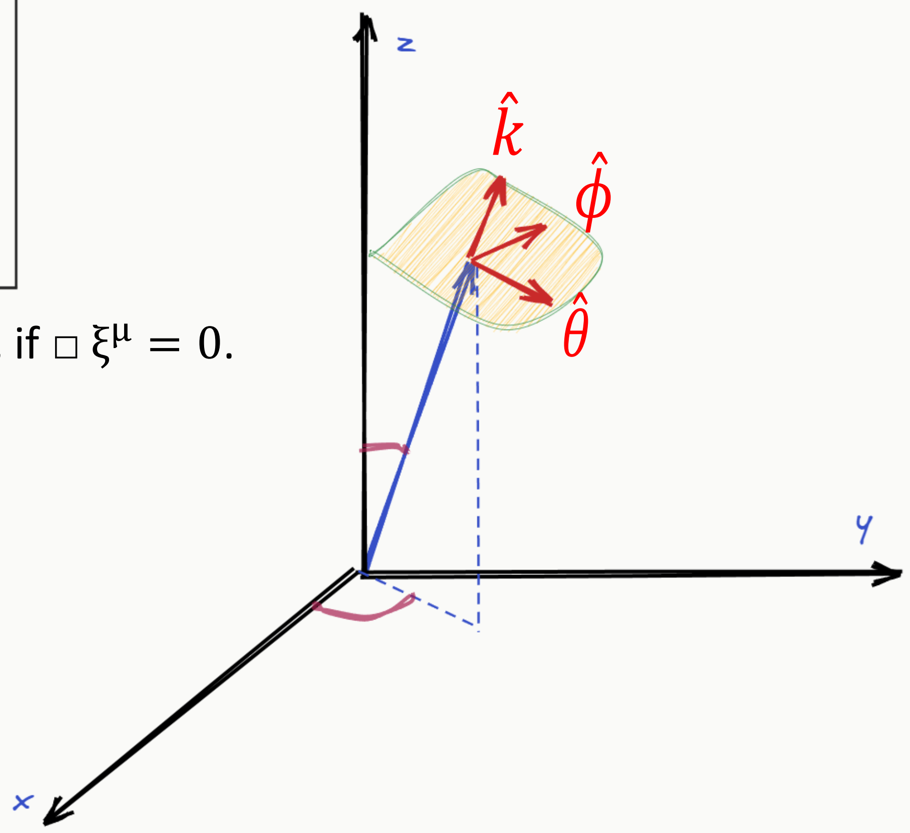
$$\partial^\mu \bar{h}'_{\mu\nu} = 0$$

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu} + \frac{1}{2}\left(\partial_\rho \partial_\mu \bar{h}^\rho{}_\nu + \partial_\rho \partial_\nu \bar{h}^\rho{}_\mu - \eta_{\mu\nu}\partial_\rho \partial_\sigma \bar{h}^{\rho\sigma}\right)$$

Vacuum wave equation for metric perturbations

The linearized Einstein equation can now be written as

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu},$$
$$\partial^\mu\bar{h}_{\mu\nu} = 0.$$



and this does not change if we make another small coordinate adjustment, if $\Box\,\xi^\mu = 0$.

In vacuum:

$$\Box\bar{h}_{\mu\nu} = 0,$$
$$\partial^\mu\bar{h}_{\mu\nu} = 0.$$

With a plane wave ansatz, $\bar{h}_{\mu\nu} = A_{\mu\nu}e^{ik_\rho x^\rho}$

$$-k_\rho k^\rho = 0 \rightarrow \omega = c|\vec{k}|$$
$$ik^\mu A_{\mu\nu} = 0.$$

- Waves propagates at the speed of light
- Polarization basis: symmetric tensors living in $\mathrm{span}\{k^\mu, (0,\hat{\theta}), (0,\hat{\phi})\}$ with $\hat{\theta}\cdot\vec{k} = \hat{\phi}\cdot\vec{k} = 0$
- Polarization degrees of freedom: 10 – 4 = 6. But still, we can adjust coordinates.

Gauge freedom: what is coordinate and what is physical?

If we do the same plane wave ansatz for the residual transformation, $\xi^\mu = iV^\mu e^{ik_\rho x^\rho}$, from $\Box\, \xi^\mu = 0$ we find the same dispersion relation of the perturbation. It follows that we can change the polarization tensor as

$$A_{\mu\nu} \to A'_{\mu\nu} = A_{\mu\nu} + \left(k_\mu V_\nu + k_\nu V_\mu - \eta_{\mu\nu} k_\rho V^\rho\right)$$

We can freely choose V^μ and impose $A_{\mu 0} = 0$, $A^\mu_\mu = 0$. This is the **transverse traceless gauge.**
It follows that the structure of the polarization tensor is

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & A_{ij} & \\ 0 & & & \end{pmatrix}$$

The block A_{ij} is transverse ($k_i A_{ij} = 0$) and traceless ($A_{ii} = 0$) and can be represented by a linear combination of the two linear polarizations:

$$\epsilon^{(+)}_{ij}(\hat{k}) = \hat{\theta}_i\hat{\theta}_j - \hat{\phi}_i\hat{\phi}_j$$

$$\epsilon^{(\times)}_{ij}(\hat{k}) = \hat{\theta}_i\hat{\phi}_j + \hat{\phi}_i\hat{\theta}_j$$

Conclusion: we have two physical polarization degrees of freedom in general relativity.

A gravitational wave as an oscillating tidal field

Imagine a set of particles with $u^\mu = (1,0,0,0)$.
In the TT gauge

$$\Gamma^\mu{}_{00} = \frac{1}{2}\eta^{\mu\lambda}\left(2\partial_0 h^{TT}_{\lambda 0} - \partial_\lambda h^{TT}_{00}\right) = 0$$

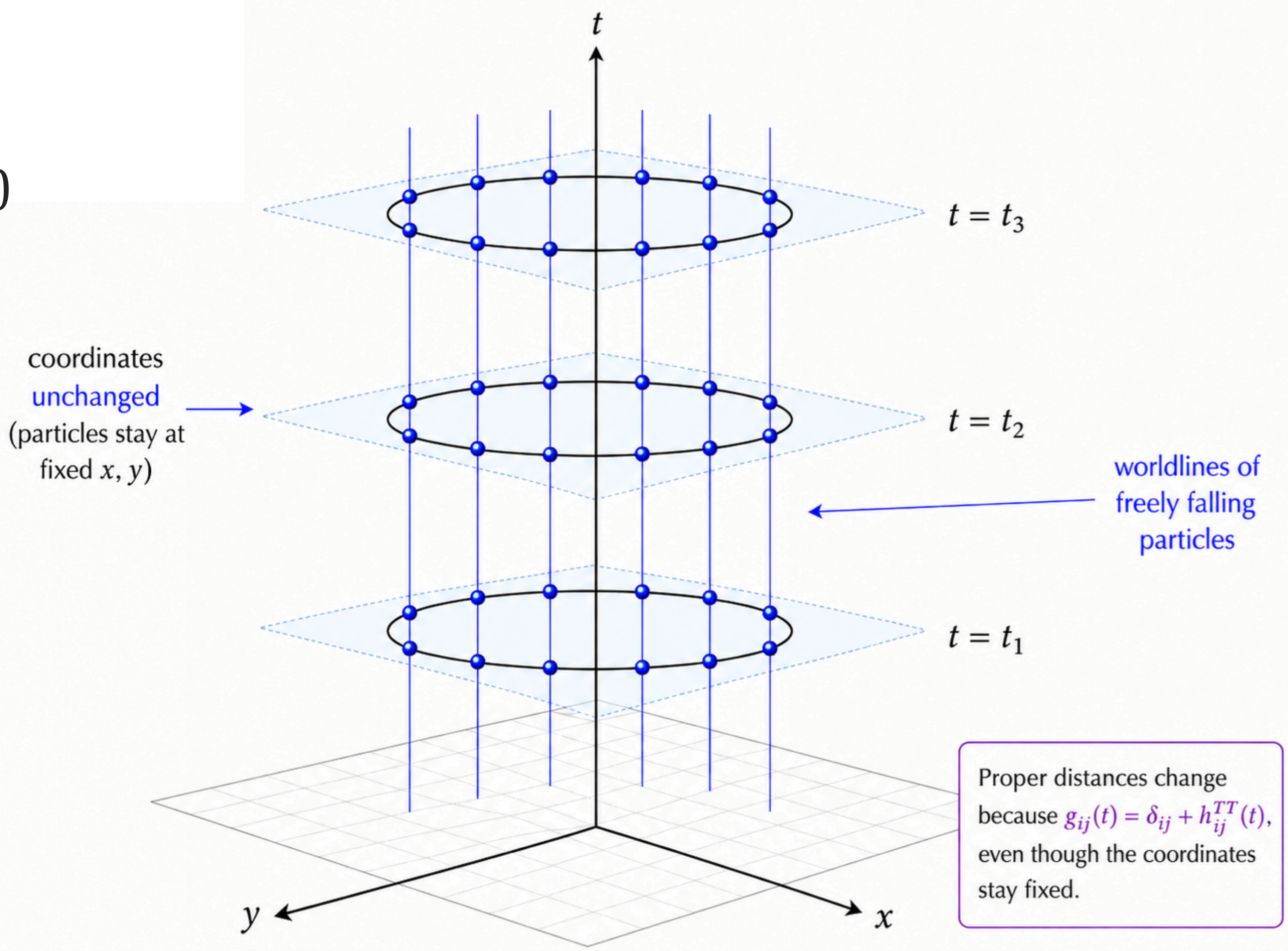
so

$$\frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho = 0$$

is satisfied (geodesic motion)

$$\frac{D^2\xi^i}{dt^2} = \frac{1}{2}\frac{\partial^2 h^{\mathrm{TT}}_{ij}}{\partial t^2}\xi^j \qquad \frac{d^2\xi^i_{\mathrm{coord}}}{dt^2} = 0$$

$$\xi^i = \left(\delta^i{}_j + \frac{1}{2}h^i{}_{j,\mathrm{TT}}\right)\xi^j_{\mathrm{coord}}$$



Strain as fractional distance change

Two test masses at $x^i = 0$ and $x^i = L\hat{e}^i$ have squared proper separation

$$L_{\text{phys}}^2 = (\delta_{ij} + h_{ij}^{\text{TT}})L^2\hat{e}^i\hat{e}^j \approx L^2(1 + h_{ij}^{\text{TT}}\hat{e}^i\hat{e}^j),$$

so $L_{\text{phys}} \approx L(1 + \frac{1}{2}h_{ij}^{\text{TT}}\hat{e}^i\hat{e}^j)$ and the *strain*

$$\boxed{\frac{\Delta L}{L} = \frac{1}{2}\,h_{ij}^{\text{TT}}\hat{e}^i\hat{e}^j\ .}$$

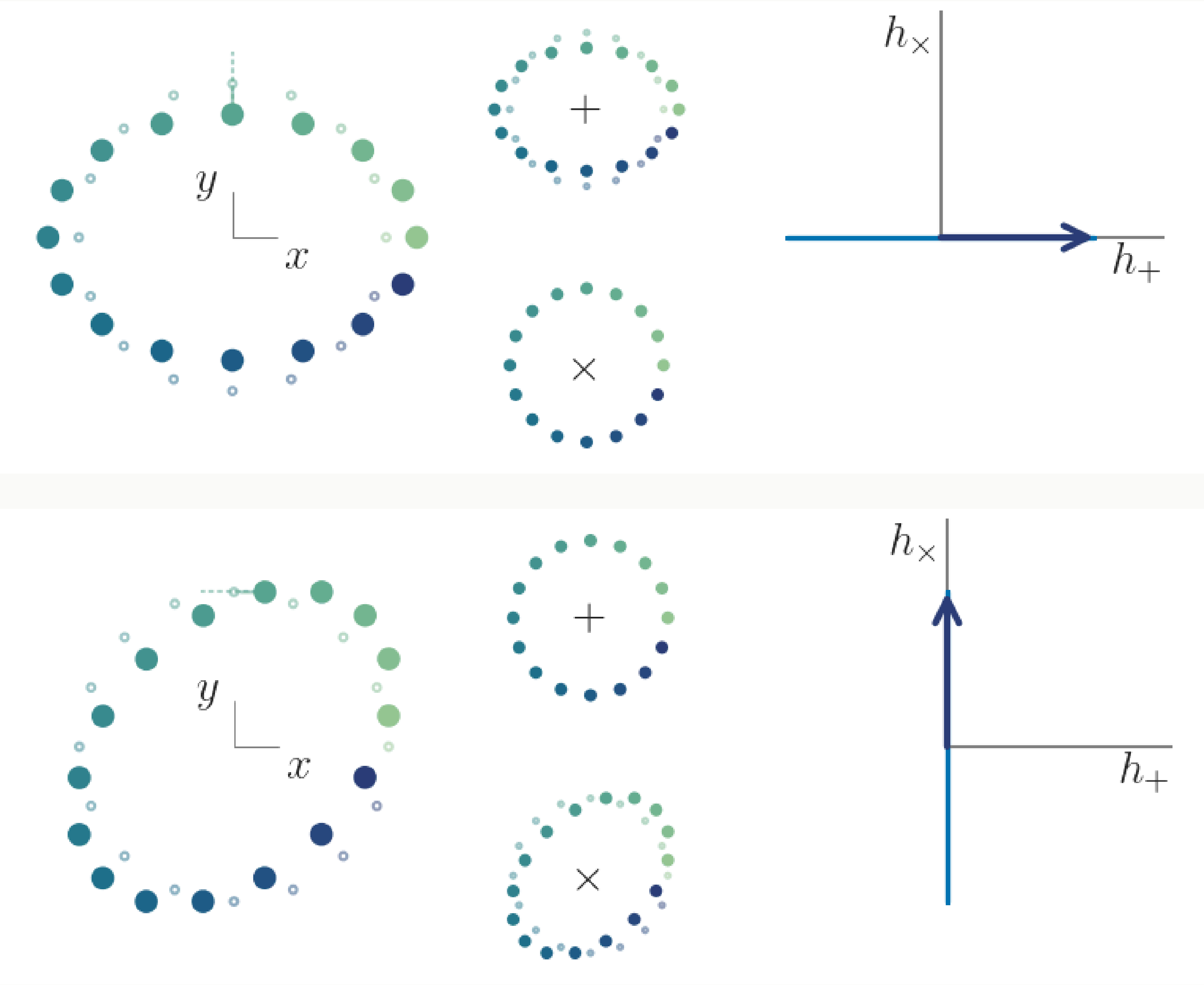
Plus and cross modes

$$h_{ij}(t) = h_+(t)\epsilon^+_{ij} + h_\times(t)\epsilon^\times_{ij}$$

A gravitational wave deforms a ring of freely falling test particles into an oscillating ellipse.

The plus mode stretches and compresses along two orthogonal axes. The cross mode does the same along axes rotated by $\pi/4$

The pattern is transverse, traceless, and quadrupolar.



Linear polarization basis

Credits: https://maxisi.github.io/gwpols/

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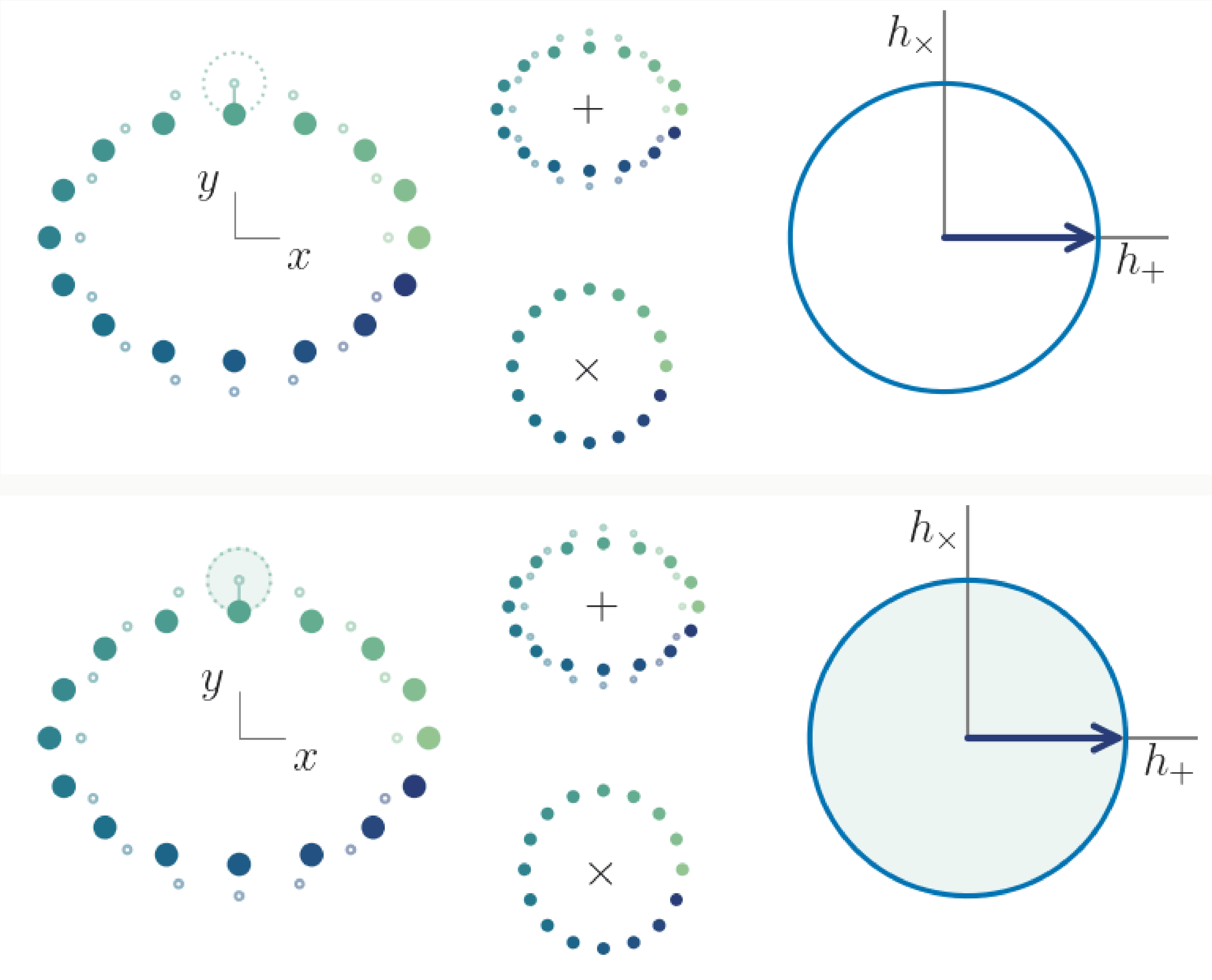

Circular polarization basis

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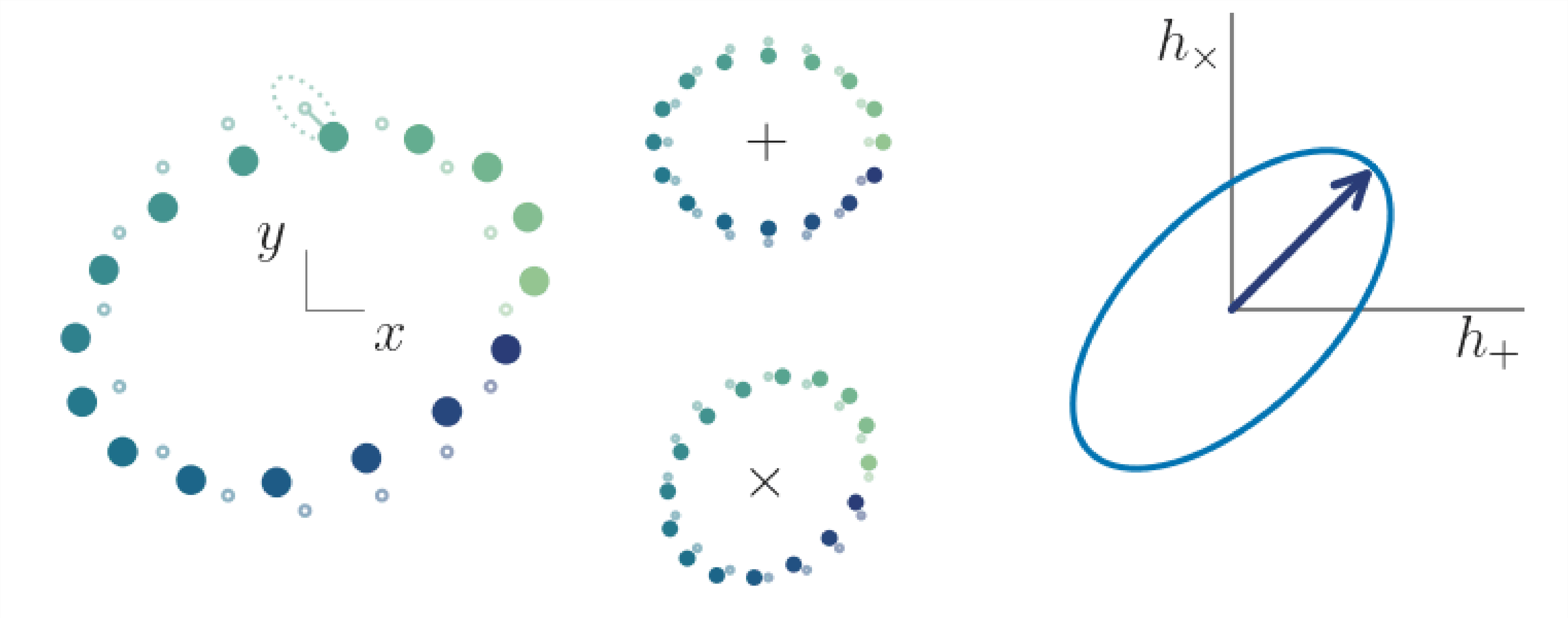
Plus and cross modes



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Why gravitational waves are weakly coupled to matter

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu},$$

$$h \sim \frac{GE}{c^4 r} \sim \frac{GM}{c^2 r} \sim \frac{r_g}{r}$$

For a stellar-mass compact binary, one may have

$$\frac{GM}{c^2} \sim 10^2\,\mathrm{km},$$

while the distance may be

$$r \sim 1\,\mathrm{Mpc} \sim 3 \times 10^{19}\,\mathrm{km}.$$

The ratio is already of order

$$\frac{GM}{c^2 r} \sim 10^{-18}.$$

After including orientation, efficiency, and the fact that only the time-varying quadrupole part radiates, typical strains become

$$h \sim 10^{-21}$$

This makes detection extremely difficult, but it also makes gravitational waves exceptionally clean probes of otherwise inaccessible regions of the Universe.

Can a gravitational wave carry energy?

A gravitational wave carries energy because it can do work on matter and because sources lose energy when they radiate.

However, the equivalence principle prevents a strict local tensorial energy density for the gravitational field.

Moreover, in the linearized theory:

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu},$$
$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

$\Rightarrow$

$$\Box \partial^\mu \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} \partial^\mu T_{\mu\nu}$$

$\Rightarrow$

$$0 = \partial^\mu T_{\mu\nu}$$

This is because we need second order corrections to give meaning to the gravitational wave energy.

Why GW energy is a second-order concept

$$G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu}[h] \quad + G^{(2)}_{\mu\nu}[h,h] + O(h^3) \quad = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$0 - \frac{1}{2}\Box\bar{h}_{\mu\nu} \quad + G^{(2)}_{\mu\nu}[h,h] + O(h^3) \quad = \frac{8\pi G}{c^4} T_{\mu\nu}$$

At the second order

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}[h,h]\right)$$

The GW energy-momentum tensor is not a local tensor of the exact metric. It is an effective, averaged object. Its definition requires a hierarchy of scales: the wave oscillates on a short scale, while the background and the detector response vary on a much longer scale.
A physically meaningful GW energy density or flux requires averaging over fast scales.

Energy flux in the TT gauge

$$G^{(1)}_{\mu\nu}[h] = 8\pi G\left(T_{\mu\nu} - t_{\mu\nu}\right), \qquad t_{\mu\nu} \equiv -\frac{1}{8\pi G}\left\langle G^{(2)}_{\mu\nu}[h]\right\rangle$$

$$\lambda_{\mathrm{GW}} \ll L_{\mathrm{avg}} \ll L_{\mathrm{background}}$$

In the TT gauge

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h^{TT}_{ij}\, \partial_\nu h^{TT}_{ij}\right\rangle$$

Polarizations contributions adds. For the energy flux we can write

$$\frac{dE}{dt\,dA} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle$$

For GW150914 ($h \sim 10^{-21}$, $f \sim 200$ Hz) this gives a peak flux $F \sim 10^{-2}\ \mathrm{W\,m^{-2}}$ at Earth — comparable in flux to the full Moon, sourced by an event lasting tens of milliseconds.

The quadrupole formula for the wave amplitude

The wave equation is solved by the retarded Green's function for the d'Alembertian:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} \, \mathrm{d}^3 x' .$$

In the slow-motion, far-field limit $(d \ll \lambda_{\mathrm{GW}} \ll r)$,

$$\bar{h}_{\mu\nu}(t,\vec{x}) \approx \frac{4G}{r} \int T_{\mu\nu}(t - r/c, \vec{x}') \, \mathrm{d}^3 x' .$$

Because only the spatial h_{ij} survives projection onto TT, and because

$$\int T^{ij} \, \mathrm{d}^3 x = \frac{1}{2} \ddot{I}^{ij}(t), \qquad I^{ij}(t) \equiv \int T^{00}(t,\vec{x}) \, x^i x^j \, \mathrm{d}^3 x ,$$

the asymptotic strain is

$$h_{ij}^{\mathrm{TT}}(t,\vec{x}) = \frac{2G}{c^4 \, r} \, \mathcal{P}_{ijkl}^{\mathrm{TT}} \, \ddot{I}_{kl}(t - r/c) .$$

In terms of the traceless quadrupole $Q_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I^k{}_k$ the formula reads the same way, since $\mathcal{P}^{\mathrm{TT}}$ removes traces.

The quadrupole luminosity formula

Insert the quadrupole strain formula into the flux formula and integrate over a sphere at infinity. The angle-averaged TT projector

$$\int \frac{\mathrm{d}\Omega}{4\pi}\, \mathcal{P}^{\mathrm{TT}}_{ijkl}(\hat{n}) \;=\; \frac{2}{5}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{2}{15}\,\delta_{ij}\delta_{kl}$$

acts on the source’s third time derivative of the quadrupole tensor, killing its trace and leaving

$$L_{GW} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\right\rangle$$

This is the Einstein quadrupole formula. Note the three time derivatives --- the source must change its quadrupole accelerationally, otherwise no power is radiated.

Why monopole and dipole radiation are absent

At leading order in linearized gravity, $\partial_\mu T^{\mu\nu} = 0$ implies:

- **Monopole.** $M = \int T^{00}\, d^3x,\ \dot{M} = 0 \Rightarrow \ddot{M} = 0.$
- **Mass dipole.** $D^i = \int T^{00} x^i\, d^3x,\ \ddot{D}^i = 0.$
- **Current dipole.** J^i is the total angular momentum, $\dot{J}^i = 0.$

So, at leading order, there is no monopole or dipole gravitational radiation.

In full GR the statement remains true, but the proof changes.

One must use the total conserved asymptotic charges (matter + gravitational field), so there are no genuine radiative $\ell = 0$ or $\ell = 1$ modes. Radiation starts at quadrupole order.

Energy balance: source loss equals wave luminosity

$$\frac{dE_{\text{source}}}{dt} = -L_{\text{GW}}$$

$$L_{\text{GW}} = \frac{G}{5c^5}\left\langle \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q^{ij}}{dt^3} \right\rangle$$

- Gravitational waves carry positive energy away from the source.
- Therefore the source mechanical (binding) energy decreases at the same rate.
- For a binary, $E_{\text{source}} \approx E_{\text{orb}} = -\frac{Gm_1 m_2}{2a}$.
- As energy is lost, E_{orb} becomes more negative, so the separation a decreases.
- A smaller orbit implies higher orbital and GW frequencies: this is the chirp.
  - Here E_{source} denotes the energy stored in the source dynamics, in practice the orbital binding energy for an inspiralling binary.

This balance law links wave emission to binary inspiral dynamics.

Circular binaries as radiating systems

1 Binary in the centre-of-mass frame

Reduced mass: $\mu = \frac{m_1 m_2}{M}$, total mass: $M = m_1 + m_2$

2 For a circular orbit of separation a :

$$\Omega^2 a^3 = GM$$

3 Time-dependent quadrupole:

$$Q_{ij} = \mu a^2 \left(\hat{e}_i \hat{e}_j - \frac{\delta_{ij}}{3} \right)$$

4 Dominant GW frequency:

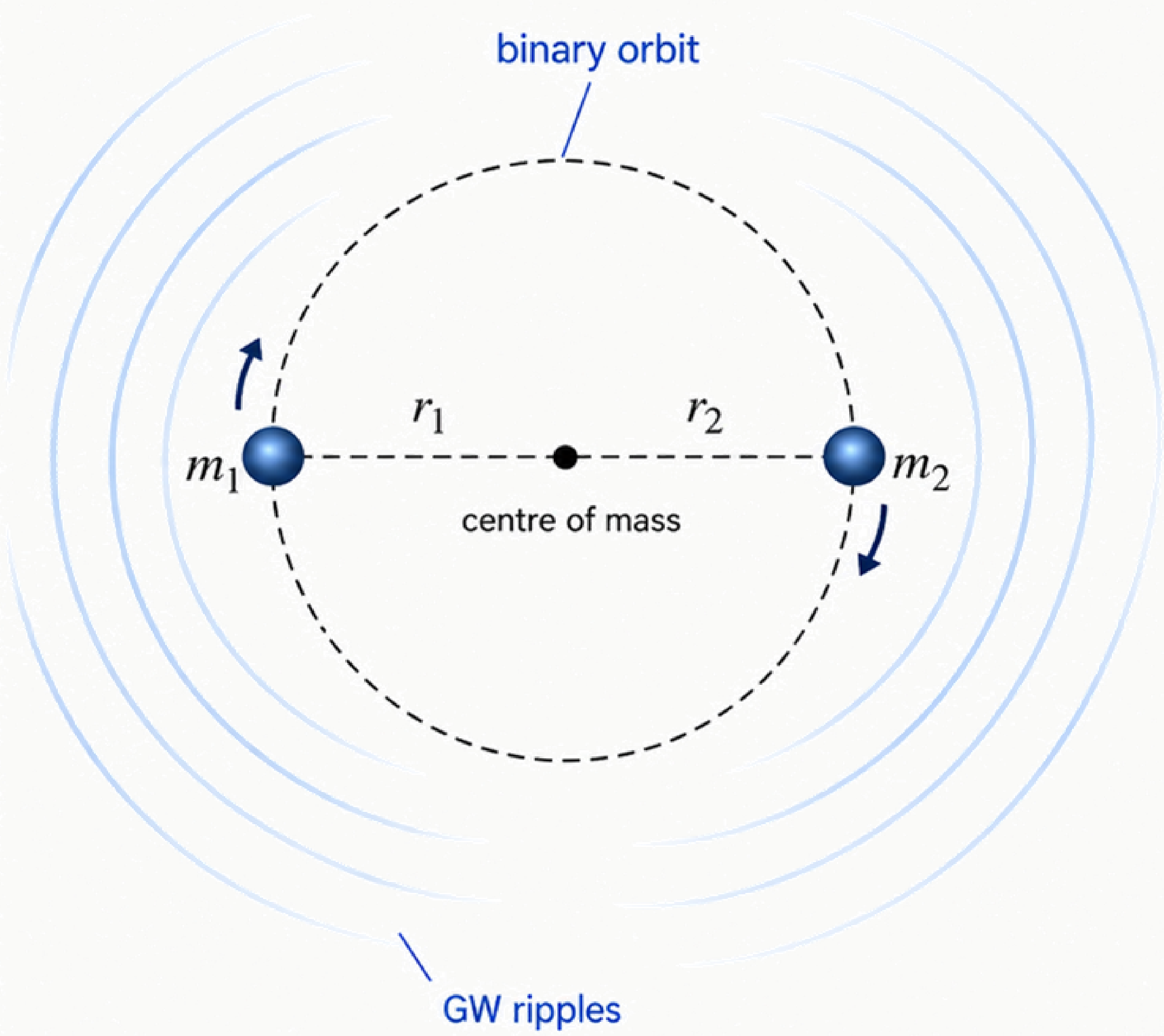
$$\omega_{\mathrm{GW}} = 2\Omega$$

5 Quadrupole luminosity (circular binary):

$$L_{\mathrm{GW}} = \frac{32}{5} \frac{G^4}{c^5} \frac{\mu^2 M^3}{a^5}$$

Key scaling: $L_{\mathrm{GW}} \propto a^{-5}$

As energy is lost ($\dot{E}_{\mathrm{orb}} = -L_{\mathrm{GW}}$),
the orbit shrinks, Ω increases, and the signal chirps.



Orbital shrinking and frequency increase

Energy balance

$$E_{\rm orb} = -\frac{G m_1 m_2}{2a} = -\frac{GM\mu}{2a}$$

$$L_{\rm GW}^{\rm circ} = \frac{32}{5}\frac{G^4}{c^5}\frac{\mu^2 M^3}{a^5}, \qquad \dot{E}_{\rm orb} = -L_{\rm GW}^{\rm circ}$$

Orbital separation

$$\dot{a} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M^2}{a^3}$$

$$\dot{a} = -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1 m_2 (m_1 + m_2)}{a^3} < 0$$

Frequency evolution

$$\Omega^2 a^3 = GM \quad \Longrightarrow \quad \frac{\dot{\Omega}}{\Omega} = -\frac{3}{2}\frac{\dot{a}}{a} > 0$$

$$\dot{\Omega} = \frac{96}{5}\frac{G^{5/3}}{c^5}\mathcal{M}^{5/3}\Omega^{11/3}$$

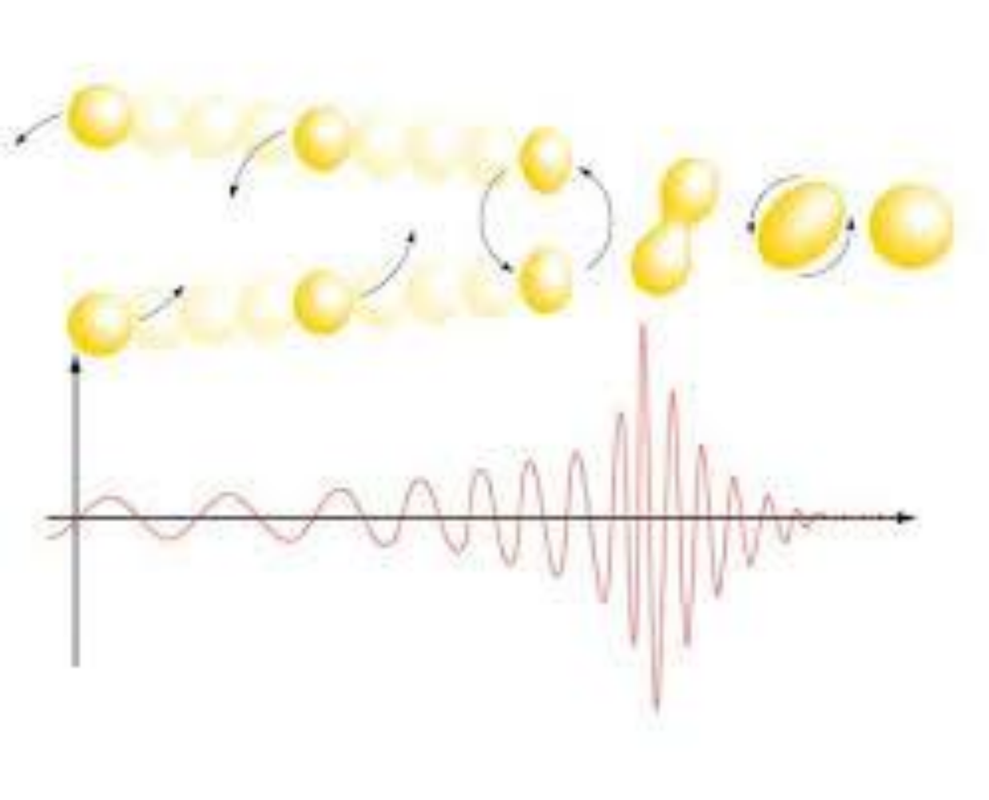
$$f_{\rm GW} = 2 f_{\rm orb} = \frac{\Omega}{\pi}, \qquad \dot{f}_{\rm GW} > 0$$

$$Q_{ij}(t) \Rightarrow L_{\rm GW} > 0 \Rightarrow \dot{E}_{\rm orb} < 0$$

$$\Rightarrow \dot{a} < 0 \Rightarrow \dot{\Omega} > 0$$

The waves carry away positive energy. A bound binary becomes more tightly bound, and the signal chirps upward in frequency.

The chirp mass and the chirp



Combine the masses into the chirp mass

$$\mathcal{M}_c \equiv \frac{(m_1\, m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \, .$$

L_{GW} and $\dot{E}_{\mathrm{orb}}$ depend on the masses only through $\mathcal{M}_c$.

$$\dot{f}_{\mathrm{GW}} = \frac{96}{5}\, \pi^{8/3} \left(\frac{G\mathcal{M}_c}{c^3} \right)^{5/3} f_{\mathrm{GW}}^{11/3} \, .$$

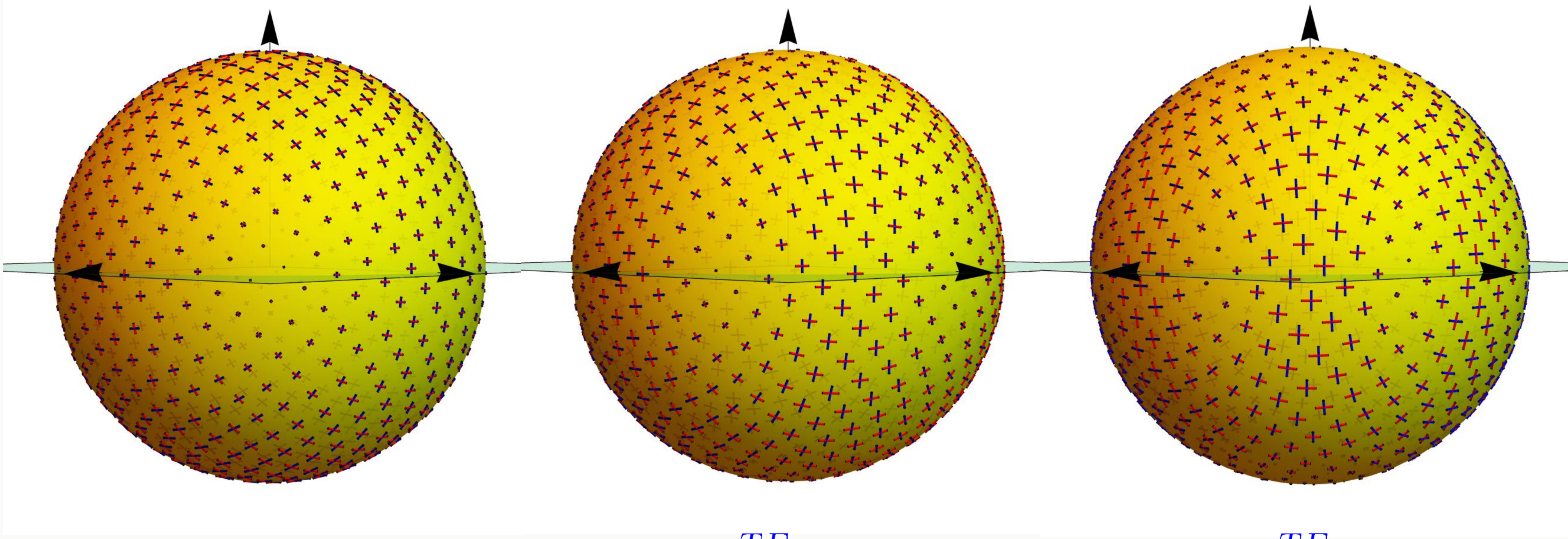
This upward sweep in frequency is the chirp.

$$\tau\,(f_{\mathrm{GW}}) = \frac{5}{256\pi^{8/3}} \frac{c^5}{(G\mathcal{M}_c)^{5/3}} f_{\mathrm{GW}}^{-8/3}$$

$$h_0 = \frac{4}{r} \left(\frac{G\mathcal{M}_c}{c^2} \right)^{5/3} \left(\frac{\pi f_{\mathrm{GW}}}{c} \right)^{2/3} \frac{1 + \cos^2 \iota}{2} \, .$$

For $\mathcal{M}_c = 30\, M_\odot$ and $f_{\mathrm{GW}} = 100$ Hz, $\tau \approx 10$ ms. The chirp mass controls the rate of sweep.

Polarization of emitted radiation



$TE_{2,2}$ $TE_{3,3}$ $TE_{4,4}$

$$\omega_{gw} = 2\omega_{orb}$$

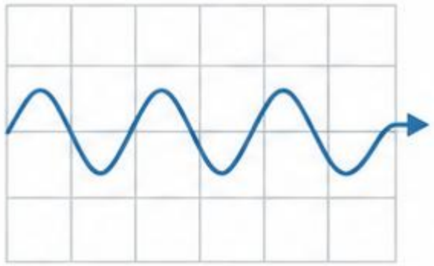
$$\omega_{gw} = \frac{3}{2}\omega_{orb}$$

$$\omega_{gw} = 4\omega_{orb}$$

From flat space to curved spacetime

Today: waves on Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$



$\eta_{\mu\nu}$

- $\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} = 0$ in vacuum
- gauge freedom $\rightarrow$ TT gauge
- two physical modes: h_+ and $h_\times$
- strain, flux, quadrupole sources

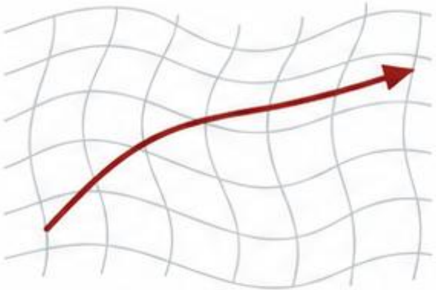
same idea

→

curved background

Next: waves on a curved background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$



$\bar{g}_{\mu\nu}(x)$

- replace ∂_μ by $\bar{\nabla}_\mu$
- curvature terms appear
- FRW: expansion and redshift
- BHs: scattering, tails, ringdown

Flat space gave us the grammar of gravitational waves.
Curved space will tell us how they travel through the Universe.