GW Detectors Principia

Block 1: Interaction of GWs with Test Masses

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Outline of Block 1

A. Setting the stage — why the gauge question matters
B. The TT frame picture
    - Test masses in TT coordinates
    - Effect of a GW on test masses
    - Light propagation and the interferometer response
C. The proper detector frame picture
    - Fermi normal coordinates and the laboratory frame
    - Newtonian force picture
D. From test masses to detector output
    - Differential strain, antenna pattern functions

*Preceded by: **G. Cella** — Theoretical aspects of GW (Wed 20, 14:00).*

Continues in: Block 2 (Thu 21, 11:05) and the rest of the school programme.

Learning goals

By the end of Block 1 you should be able to:

- explain why the same physics looks different in the TT frame and in the proper detector frame;
- compute the proper-distance change between free test masses induced by a GW;
- describe the interferometer response in TT gauge, including the $\mathrm{sinc}(\omega_{\mathrm{gw}} L/c)$ factor;
- derive and use the Newtonian force picture in the proper detector frame;
- write down the detector output $h(t) = F_+ h_+ + F_\times h_\times$ and understand its geometrical meaning.

A. Gauge freedom: physics invariant, description is not

- The physics must be invariant under coordinate transformations.
- But the *language* we use to describe GWs and detectors depends on the frame we choose.
- Two natural but different choices:
  - TT frame: GWs have the simplest form (two independent components, h_+ and $h_\times$).
  - Proper detector frame: the apparatus is described the way an experimentalist would describe it.
- Working through both clarifies the physical meaning of the experiment.

Key tools we will use

Geodesic equation, geodesic deviation, local inertial frames (Riemann normal coordinates), free-falling frames (Fermi normal coordinates).

Recap — TT gauge (stated, not derived)

Starting from the Lorenz gauge $\partial^\beta \bar{h}_{\alpha\beta} = 0$, the residual gauge freedom allows us to impose

$$h_{0\alpha} = 0, \qquad h^i{}_i = 0, \qquad \partial^j h_{ij} = 0.$$

In vacuum the field equations reduce to

$$\Box h_{ij}^{\mathrm{TT}} = 0,$$

with only two independent components h_+ and $h_\times$ and $\bar{h}_{ij} = h_{ij}$ (the perturbation is already traceless).

- For a plane wave propagating along $\hat{z}$:

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega_{\mathrm{gw}}(t - z/c)].$$

→ *Full derivation:* ***G. Cella*** *— Theoretical aspects of GW (Wed 20, 14:00).*

Recap — Riemann tensor of a linear GW

In linearized theory the Riemann tensor is *gauge invariant*, hence we can evaluate it in any convenient frame. In the TT gauge:

$$R_{i0j0} = -\frac{1}{2c^2}\,\ddot{h}_{ij}^{\mathrm{TT}}.$$

- All other components of R relevant for us follow from this and from the symmetries of the Riemann tensor.
- In particular, $R^i{}_{0j0} = \eta^{ik} R_{k0j0} = R_{i0j0}$.
- This expression will feed directly into the geodesic deviation equation on the next slide.

Recap — geodesic and geodesic-deviation equations

Geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0.$$

Geodesic deviation (two neighbouring free particles, separation ξ^μ)

$$\frac{D^2 \xi^\mu}{D\tau^2} = -\, R^\mu{}_{\alpha\beta\gamma}\, u^\alpha\, \xi^\beta\, u^\gamma .$$

For a test mass initially at rest $(u^\mu \simeq (c, \vec{0}))$ and at lowest order in h:

$$\ddot{\xi}^{\,i} \;=\; -\, c^2 R^i{}_{0j0}\, \xi^j \;=\; \tfrac{1}{2}\, \ddot{h}^{\mathrm{TT}}_{ij}\, \xi^j .$$

Roadmap: two complementary pictures

TT frame
simple wave, abstract lab

⟷ same physics
coincide in long-λ limit
$\omega_{gw} L/c \ll 1$

Proper detector frame
Newtonian-looking lab

Coordinates stretch, test masses stay at rest by definition

Rigid ruler, test masses driven by a Newtonian tidal force

- We will use **TT** when dealing with the wave itself and with light propagation along the arms.
- We will use the **proper detector frame** when reasoning about mirrors, suspensions and feedback.

B. Test mass at rest in the TT frame

Consider a free test mass at rest at $\tau = 0$, $u^\mu\big|_{\tau=0} = (c, \vec{0})$. The geodesic equation gives

$$\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\,\Gamma^i_{00}(u^0)^2\Big|_{\tau=0}$$

With $\Gamma^i_{00} = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00})$, and using the TT conditions $h_{00} = h_{0i} = 0$:

$$\Gamma^i{}_{00} = 0 \implies \left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = 0$$

Since both u^i and $du^i/d\tau$ vanish initially, the coordinate position remains unchanged *at all times*.

In the TT frame, free test masses initially at rest remain at rest (in *coordinates*) even after the passage of the wave.

Physical meaning — coordinates stretch with the wave

- In TT gauge, the *coordinates* are physically defined using the free test masses themselves as markers.
- Coordinates stretch in response to the wave, in such a way that the coordinate position of a free test mass does not change.
- Therefore the *coordinate separation* of free test masses is also constant.
- This is a very clean example of a GR principle: physical effects live in invariant quantities (proper distances, proper times), **not** in coordinates.

Warning

"Coordinates stay put" $\neq$ "nothing happens". The observable quantity is the **proper distance** between test masses, which *does* change.

Proper distance between free test masses

Two test masses at TT-coordinate separation ξ^i. The proper distance at time t is

$$s(t) = \int_0^L \sqrt{g_{ij}\,dx^i\,dx^j} \;\simeq\; L\left[1 + \tfrac{1}{2}\,h_{ij}^{\mathrm{TT}}\,\hat{n}^i\hat{n}^j\right],$$

where $\hat{n}^i = \xi^i/L$ is the unit vector along the separation.

- The fractional change is
$$\frac{\Delta s}{L} = \frac{1}{2}\,h_{ij}^{\mathrm{TT}}\,\hat{n}^i\hat{n}^j.$$
- This matches the geodesic-deviation result: integrating $\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}\xi^j$ and using $\xi^i \simeq L\hat{n}^i$ at zeroth order gives $\Delta\xi^i = \frac{1}{2}h_{ij}^{\mathrm{TT}}\,\xi^j$.

Effect on a ring of test masses — plus polarization

h_+ only, wave along $\hat{z}$:

$$h_{ij}^{\mathrm{TT}} = h_+(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Displacement of a particle at $\xi^i = (x_0, y_0, 0)$:

$$\Delta x = \tfrac{1}{2}\, h_+(t)\, x_0, \qquad \Delta y = -\tfrac{1}{2}\, h_+(t)\, y_0.$$

Anti-correlated stretching along x and y.

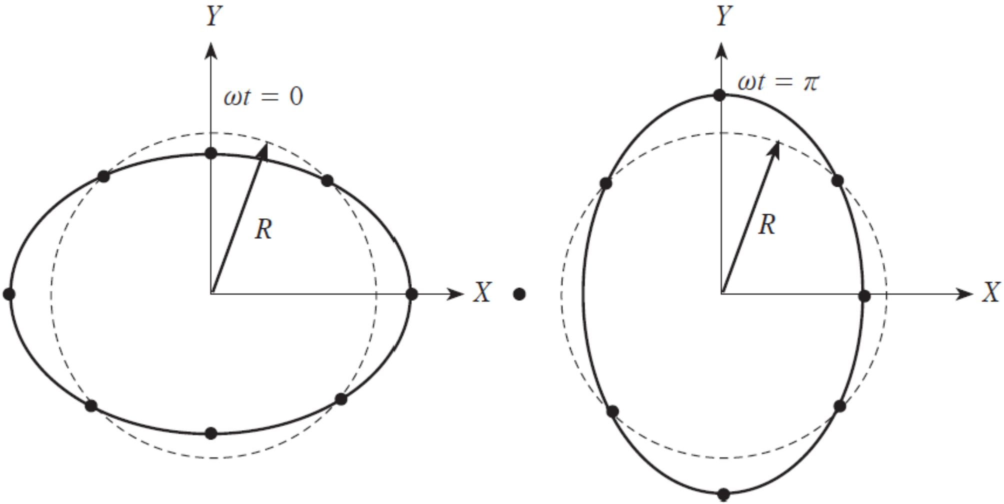

Figure: Ring deformation, h_+ polarization.

Effect on a ring of test masses — cross polarization

$h_\times$ only, wave along $\hat{z}$:

$$h_{ij}^{\mathrm{TT}} = h_\times(t) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Displacement of a particle at $\xi^i = (x_0, y_0, 0)$:

$$\Delta x = \tfrac{1}{2}\, h_\times(t)\, y_0, \qquad \Delta y = \tfrac{1}{2}\, h_\times(t)\, x_0.$$

Same pattern as h_+ but rotated by 45°.

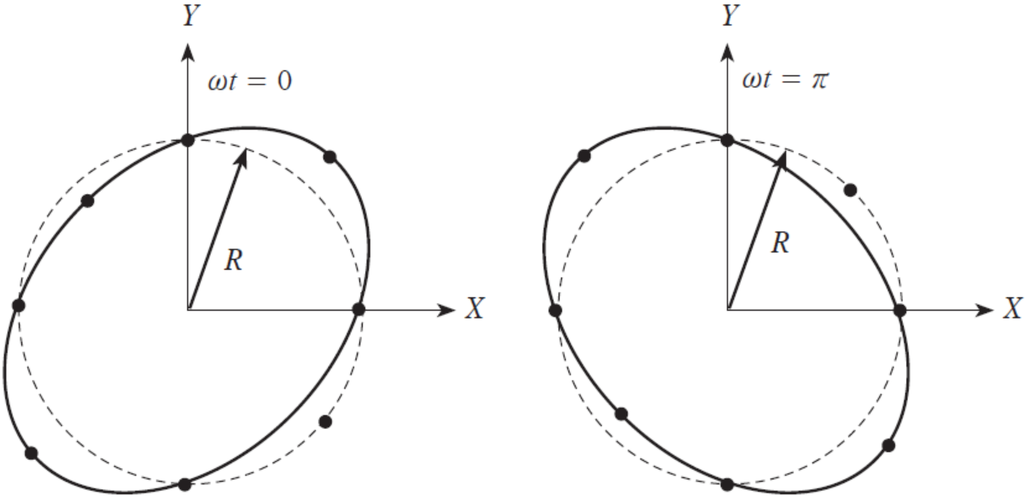

Figure: Ring deformation, $h_\times$ polarization.

Light propagation along one arm — setup

Arm along $\hat{x}$, length L. Wave h_+ propagating along $\hat{z}$, $h_{xx}^{\mathrm{TT}}(t) = h(t)$.

- The metric in TT gauge along the arm reads

$$ds^2 = -c^2\,dt^2 + [1 + h(t)]\,dx^2.$$

- Null geodesics ($ds^2 = 0$): $c\,dt = \sqrt{1 + h(t)}\,dx$, hence to first order in h

$$dt = \frac{dx}{c}\left[1 + \tfrac{1}{2}h(t)\right].$$

- Coordinate position of the end mirrors is constant in TT (previous result): the integration limits are 0 and L.

Round-trip time — the *sinc* factor

Forward trip (from 0 to L starting at time t_0) plus return (from L to 0). For a monochromatic wave $h(t) = h_0 \cos(\omega_{\rm gw} t)$:

$$t_{\rm rt}(t_0) - \frac{2L}{c} = \frac{L}{c}\, h_0 \,\mathrm{sinc}\left(\frac{\omega_{\rm gw} L}{c}\right) \cos\left[\omega_{\rm gw}\left(t_0 + \tfrac{L}{c}\right)\right],$$

where $\mathrm{sinc}(x) \equiv \sin x/x$.

- Light travel time acquires a correction *proportional to* h_0.
- The frequency-dependent factor $\mathrm{sinc}(\omega_{\rm gw} L/c)$ encodes the finite-speed propagation of the photon inside the arm.
- For $\omega_{\rm gw} L/c \to 0$, $\mathrm{sinc} \to 1$: one recovers the naive "arm stretches by $hL/2$, photon takes longer by hL/c" picture.

$\to$ *In Block 2 the sinc is superseded by the Fabry-Perot cavity pole, which sets the effective bandwidth once arm cavities are introduced.*

Regimes of the sinc factor

- **Long-wavelength limit** $\lambda_{\rm gw} \gg L$: $\mathrm{sinc} \to 1$. The arm behaves as if it were instantaneously stretched.
- **Short-wavelength limit** $\lambda_{\rm gw} \ll L$: $\mathrm{sinc} \to 0$. The GW oscillates many times during one photon transit and averages to zero.
- Zero crossings at $\omega_{\rm gw} L/c = n\pi \Rightarrow f_{\rm gw} = n\,c/(2L)$.

Numbers:

- Virgo, $L = 3$ km $\Rightarrow$ first zero at ~ 50 kHz.
- ET, $L = 10$ km $\Rightarrow$ first zero at ~ 15 kHz.
- All current detectors operate in the long-wavelength regime.

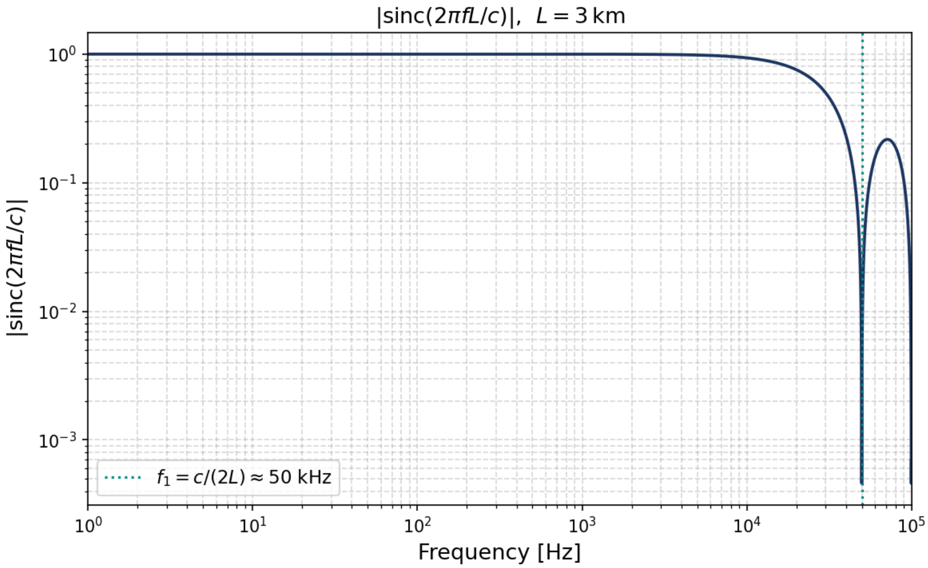

Figure: $\mathrm{sinc}(\omega L/c)$ vs frequency (Virgo: 3 Km arm length).

Audio sidebands at the output

Laser field at the input: $A_{\rm in}(t) = A_0\, e^{-i\omega_L t}$. After one round trip the field is $A_{\rm in}(t - t_{\rm rt})$. Expanding to first order in h_0:

$$A_{\rm out}(t) = A_0\, e^{-i\omega_L(t-2L/c)} \left[1 + \tfrac{i}{2}\, h_0\, k_L L\, \mathrm{sinc}\!\left(\frac{\omega_{\rm gw} L}{c}\right)\!\left(e^{i\omega_{\rm gw}(t-L/c)} + e^{-i\omega_{\rm gw}(t-L/c)}\right)\right].$$

- Two audio sidebands at $\omega_L \pm \omega_{\rm gw}$.
- Sideband amplitude $\propto k_L L\, h_0 = (2\pi L/\lambda_L)\, h_0$ — *linear* in L.
- Reading out at the beat between carrier and sidebands gives a signal $\propto h_0$.

C. Why experimentalists use a different frame

- In a lab, positions are marked with *rigid rulers* and an arbitrarily chosen origin.
- In such a frame, we *expect* free test masses to move under the action of a GW, relative to the rigid ruler.
- This is the opposite of what we found in the TT frame.
- The frame that realizes this intuition is the proper detector frame.
- For a free-falling lab, this is the Fermi normal frame built around the central worldline.
- For an Earth-bound lab, add the local gravitational acceleration $\mathbf{g}$ and Earth rotation $\boldsymbol{\Omega}$.

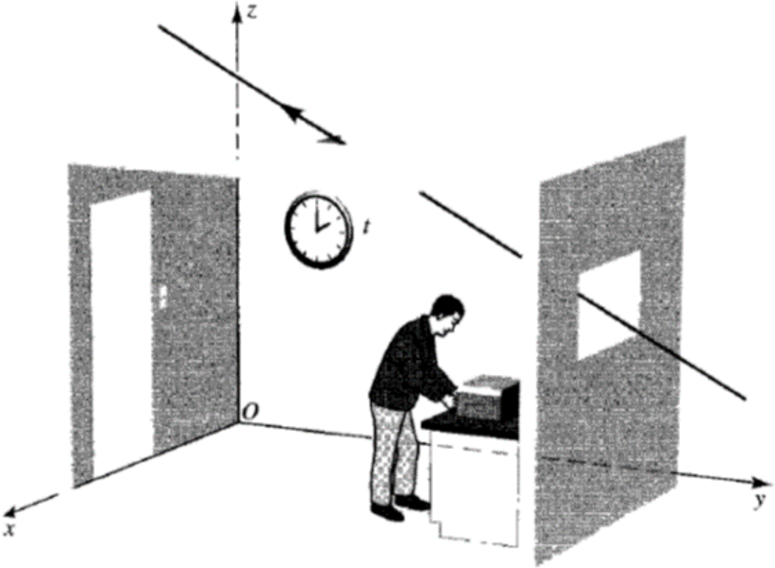

Figure: Rigid laboratory frame.

Fermi normal coordinates — free-falling lab

Around a central free-falling worldline, in a tube of size $r \ll L_B$ (curvature scale):

$$
\begin{aligned}
ds^2 = -c^2 dt^2 &\left[1 + R_{0i0j}\, x^i x^j\right] \\
&- 2\, c\, dt\, dx^i \left[\tfrac{2}{3} R_{0jik}\, x^j x^k\right] \\
&+ dx^i dx^j \left[\delta_{ij} - \tfrac{1}{3} R_{ikjl}\, x^k x^l\right] .
\end{aligned}
$$

- At $r = 0$: flat Minkowski metric, $\Gamma^{\alpha}_{\beta\gamma} = 0$.
- Corrections scale as $(r/L_B)^2$, with $R \sim 1/L_B^2$.
- The Riemann tensor contains *both* the slowly-varying tidal background (Earth, Moon, ...) *and* the GW contribution.

The real detector is *not* in free fall:

- supported against gravity: $\mathbf{a} = -\mathbf{g}$;
- rotating with the Earth: angular velocity $\boldsymbol{\Omega}$.

Performing the coordinate transformation from the local free-falling frame:

$$
\begin{aligned}
ds^2 = -c^2 dt^2 \left[1 + \tfrac{2}{c^2}\,\mathbf{a}\cdot\mathbf{x} + \tfrac{1}{c^4}(\mathbf{a}\cdot\mathbf{x})^2 - \tfrac{1}{c^2}(\boldsymbol{\Omega}\times\mathbf{x})^2 + R_{0i0j}\,x^i x^j\right] \\
+ 2\,c\,dt\,dx^i \left[\tfrac{1}{c}\,\epsilon_{ijk}\,\Omega^j x^k - \tfrac{2}{3} R_{0jik}\,x^j x^k\right] \\
+ dx^i dx^j \left[\delta_{ij} - \tfrac{1}{3} R_{ikjl}\,x^k x^l\right].
\end{aligned}
$$

This is the metric in the proper detector frame.

Hierarchy of corrections in r/L_B

- **Zeroth order** (r^0): flat Minkowski metric. Newtonian physics on flat spacetime.
- **First order** (r^1): terms $\propto (\mathbf{a} \cdot \mathbf{x})$ and $(\boldsymbol{\Omega} \times \mathbf{x})$ reproduce Newtonian gravity, centrifugal and Coriolis forces.
- **Second order** (r^2): Riemann-tensor terms. Slowly-varying tidal effects *plus* the GW.

Why detectors can see GWs at all

The tidal (second-order) terms from static gravity are enormous compared to those from GWs. **What saves us** is that GWs oscillate at high frequency, 10 Hz–10 kHz, far above the time scales of static tides and quasi-static disturbances.

Newtonian-force picture

Within the GW band, all slowly-varying effects are compensated (gravity by suspensions, rotation is DC, tides are below the band). Only the GW part of the Riemann tensor survives, and $R^i{}_{0j0} = -\frac{1}{2c^2}\,\ddot{h}_{ij}^{\mathrm{TT}}$. Geodesic deviation becomes

$$\ddot{\xi}^{\,i} = -c^2 R^i{}_{0j0}\,\xi^j = \frac{1}{2}\,\ddot{h}_{ij}^{\mathrm{TT}}\,\xi^j.$$

Newtonian force per unit mass

$$F^i = \frac{m}{2}\,\ddot{h}_{ij}^{\mathrm{TT}}\,\xi^j.$$

- The dynamics reduces to Newtonian mechanics of a driven system. GR now enters only through the prescription of this tidal force.
- This is the picture we use whenever we discuss mirrors, suspensions, feedback loops.

Ring of test masses — revisited in the force picture

Force on a mass at $\xi^i = (x_0, y_0, 0)$ for h_+ polarization:

$$F_x = \frac{m}{2}\,\ddot{h}_+(t)\,x_0, \qquad F_y = -\frac{m}{2}\,\ddot{h}_+(t)\,y_0.$$

- Quadrupolar force pattern.
- Identical response to the TT calculation (after integration in time).
- For $h_\times$: rotated 45°, as before.

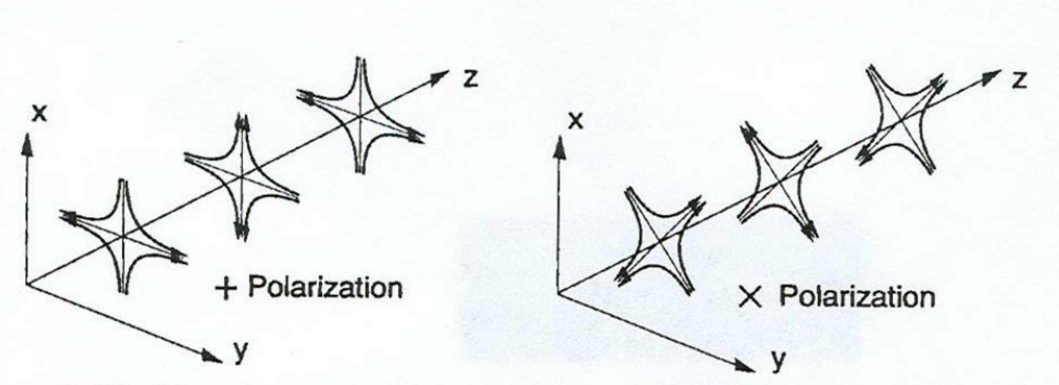

Figure: Quadrupolar force pattern.

Validity and limits of the two pictures

- The proper detector frame is valid only for $r \ll L_B$, with $L_B \sim \lambda_{\rm gw}$ for the GW contribution.
- For typical detectors, L is a few kilometres and $\lambda_{\rm gw} \gtrsim 30$ km: the proper detector frame is safe throughout the band.
- In this regime, the TT round-trip analysis gives $\mathrm{sinc} \simeq 1$, matching the Newtonian force picture exactly.
- For $\lambda_{\rm gw} \lesssim L$ (e.g. LISA, or pulsar timing), only the **full TT treatment** remains reliable.

Rule of thumb

Use the proper detector frame for noise modelling and control. Use TT gauge for light propagation and for long arms.

D. L-shaped detector — differential response

Two orthogonal arms along $\hat{n}_x = \hat{x}$ and $\hat{n}_y = \hat{y}$, common corner mass, end masses at distance L.

- Proper-length change of each arm (from slide 11):

$$\frac{\Delta L_x}{L} = \tfrac{1}{2} h_{ij}^{\mathrm{TT}} \hat{x}^i \hat{x}^j = \tfrac{1}{2} h_{xx}^{\mathrm{TT}}, \qquad \frac{\Delta L_y}{L} = \tfrac{1}{2} h_{yy}^{\mathrm{TT}}.$$

- For a wave along $\hat{z}$, $h_{yy}^{\mathrm{TT}} = -h_{xx}^{\mathrm{TT}}$ (traceless), so

$$\Delta L_x - \Delta L_y = L\, h_{xx}^{\mathrm{TT}}.$$

- The differential configuration doubles the signal and common-mode rejects many disturbances.

Strain — the dimensionless observable

Definition

$$h(t) \equiv \frac{\Delta L_x(t) - \Delta L_y(t)}{L}.$$

- Dimensionless by construction.
- Order of magnitude for detectable astrophysical sources: $h \sim 10^{-21}$.
- For $L = 3$ km: differential arm length change $\Delta L \sim 3 \times 10^{-18}$ m (less than 1/1000 of the classical proton radius).
- Whole detector design starts from this number.

All the engineering challenge is in turning a fractional length change of 10^{-21} into a measurable optical signal.

Detector tensor and antenna pattern functions

The differential output of an L-shaped detector can be written as

$$h(t) = D^{ij}\, h^{\mathrm{TT}}_{ij}(t), \qquad D^{ij} = \tfrac{1}{2}\left(\hat{n}_x^{\,i}\hat{n}_x^{j} - \hat{n}_y^{\,i}\hat{n}_y^{j}\right).$$

For a wave from sky direction (θ, ϕ) and polarization angle ψ one defines the antenna pattern functions

$$h(t) = F_+(\theta, \phi, \psi)\, h_+(t) + F_\times(\theta, \phi, \psi)\, h_\times(t).$$

- F_+, $F_\times$: geometric projection of the polarization basis onto the detector tensor.
- Quadrupolar in (θ, ϕ): four maxima, four blind spots.
- Time-dependent for long-duration signals, through the rotation of the Earth.

The fundamental detector output

$$\boxed{h(t) = F_+(\theta, \phi, \psi)\, h_+(t) + F_\times(\theta, \phi, \psi)\, h_\times(t)}$$

- A *single* detector measures one linear combination of the two polarizations, weighted by geometry.
- This is why:
  - we cannot disambiguate polarization with a single detector;
  - sky localization requires a network;
  - detector orientation matters for source-population studies.
- Figure of merit for the detector network: how uniformly can we cover the sky.

Geometry — orientation, sky location, polarization

- (θ, ϕ): source direction in detector-fixed frame.
- ψ: polarization angle, rotation of the h_+ / $h_\times$ basis in the plane orthogonal to the propagation.
- $F_+^2 + F_\times^2$: overall directional sensitivity of the detector.

Best response: source overhead (perpendicular to the detector plane).

Blind spots: source in the plane of the arms, along the bisectors.

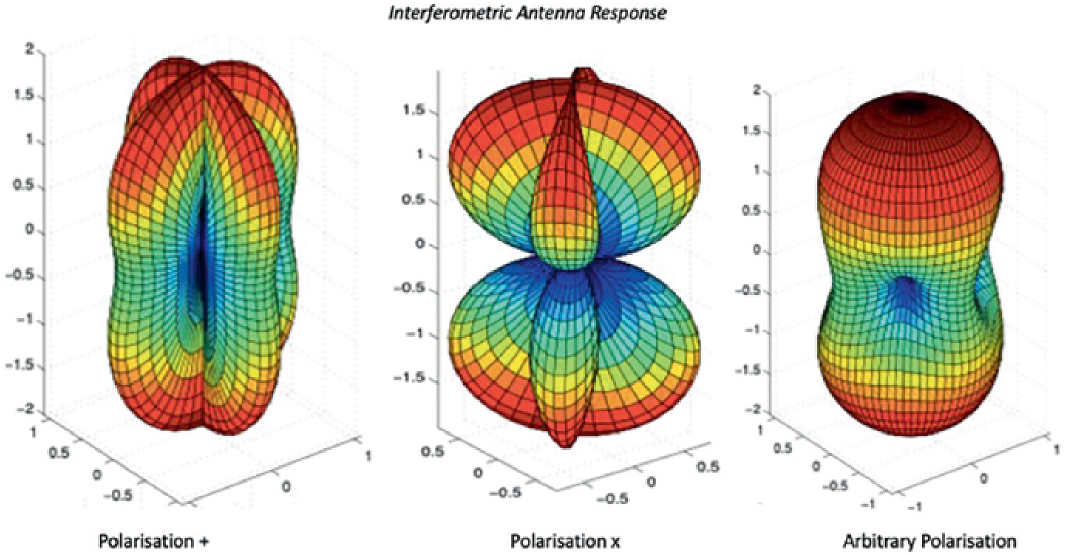

Figure: Antenna pattern of an L-shaped detector.

Bridge to Block 2

- We now know:
  - what $h(t)$ is;
  - how the detector geometry projects the two GW polarizations onto a single scalar observable;
  - how light propagation maps $h(t)$ into an optical signal (audio sidebands, sinc factor).
- Block 2 is about the instrument that *reads* this signal out of the noise floor:
  - Michelson + Fabry-Perot + recycling;
  - fundamental and technical noises;
  - the sensitivity curve as a map of the detector;
  - pointers to advanced techniques (squeezing, signal recycling, ...).
- Dedicated school lectures that follow:
  - **A. Perreca** — Optical configurations (Fri 22 & Sat 23)
  - **D. Bersanetti** — Interferometer controls (Sat 23)
  - **F. Sorrentino** — Squeezing (Tue 26)

Block 1 — Summary

- **Gauge matters for description, not for physics.** TT frame and proper detector frame give the same answers (in the long-λ limit).
- **TT frame**: free test masses stay at rest in coordinates, *proper* distances change.
- **Light propagation in TT**: round-trip time acquires a correction $\propto h\, L\, \mathrm{sinc}(\omega L/c)$, producing audio sidebands.
- **Proper detector frame**: tidal Newtonian force $F_i = \frac{m}{2}\ddot{h}_{ij}\xi^j$. Convenient for noise modelling.
- **Detector output**: $h(t) = F_+ h_+ + F_\times h_\times$.

Key equations to leave with

1. Proper-distance change: $\dfrac{\Delta s}{L} = \frac{1}{2}\, h_{ij}^{\mathrm{TT}}\, \hat{n}^i \hat{n}^j$.
2. Newtonian tidal force: $F^i = \frac{m}{2}\, \ddot{h}_{ij}^{\mathrm{TT}}\, \xi^j$.
3. Light round-trip correction: $\Delta t_{\mathrm{rt}} = \frac{L}{c}\, h\, \mathrm{sinc}\left(\frac{\omega_{\mathrm{gw}} L}{c}\right)$.
4. Detector output: $h(t) = F_+ h_+ + F_\times h_\times$.

Take-away number

For $h \sim 10^{-21}$ and $L \sim 3$ km: $\Delta L \sim 3 \times 10^{-18}$ m.

[1] Michele Maggiore. *Gravitational Waves. Vol. 1: Theory and Experiments*. Oxford: Oxford University Press, 2007. ISBN: 978-0-19-857074-5. DOI: 10.1093/acprof:oso/9780198570745.001.0001.

[2] Jolien D. E. Creighton and Warren G. Anderson. *Gravitational-Wave Physics and Astronomy: An Introduction to Theory, Experiment and Data Analysis*. Weinheim: Wiley-VCH, 2011. ISBN: 978-3-527-40886-3. DOI: 10.1002/9783527636037.

[3] Peter R. Saulson. *Fundamentals of Interferometric Gravitational Wave Detectors*. 2nd ed. Singapore: World Scientific, 2017. ISBN: 978-981-314-307-4. DOI: 10.1142/9book.

[4] Éanna É. Flanagan and Scott A. Hughes. “The basics of gravitational wave theory”. In: *New Journal of Physics* 7 (2005), p. 204. DOI: 10.1088/1367-2630/7/1/204. arXiv: gr-qc/0501041.

End of Block 1

Questions?

Next: Block 2 — Experimental fundamentals

Michelson, Fabry-Perot, recycling, noise budget, reading a sensitivity curve.