GW Detectors Principia

Exercise Solutions

STGWD 2026 — PhD International School on
Technologies in Gravitational Waves Detection

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May 2026

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Preface

How to use this document. Each solution outline gives the key idea, the main steps, and the final result. Full algebraic detail is intentionally omitted: the goal is to let you check that your reasoning is on the right track, not to replace the calculation. Work through each exercise before consulting the solution.

The exercises are numbered as in the lecture notes. References to equations in the notes are written as (Notes, Eq. X).

Chapter 1

Block 1 — Interaction of GWs with Test Masses

1. **Christoffel symbols in TT gauge.**

Solution outline.

Key idea. In TT gauge $h_{0\alpha} = 0$ identically everywhere (not just on one worldline), and h_{ij}^{TT} is purely spatial. The Christoffel formula $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta})$ reduces to $\eta^{\mu\nu}$ at first order in h.

Step 1: Γ^i_{00}**.** The three metric derivatives involve $\partial g_{0\nu}$. Since $h_{0\alpha} = 0$, we have $\partial g_{0\nu} = 0$, so $\Gamma^i_{00} = 0$. This is the result used to show that a free test mass initially at rest in TT coordinates stays at rest.

Step 2: Γ^0_{ij}**.** The relevant combination is

$$\Gamma^0_{ij} = \tfrac{1}{2}\eta^{00}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}).$$

The first two terms vanish by $h_{0\alpha} = 0$; the third gives

$$\Gamma^0_{ij} = -\tfrac{1}{2}\,(-1)\,\partial_0 h_{ij}^{\mathrm{TT}} = \frac{1}{2c}\,\dot{h}_{ij}^{\mathrm{TT}}.$$

(Here $\partial_0 = \partial/(c\,\partial t)$ with the $\eta^{00} = -1$ convention.)

Step 3: Γ^i_{0j}**.** The relevant combination is

$$\Gamma^i_{0j} = \tfrac{1}{2}\eta^{ik}(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{0j}).$$

The last two terms vanish by $h_{0\alpha} = 0$; the first gives

$$\Gamma^i_{0j} = \tfrac{1}{2}\,\delta^{ik}\,\dot{h}_{kj}^{\mathrm{TT}}/c = \frac{1}{2c}\,\dot{h}^{i\,\mathrm{TT}}_{\;j}.$$

All results are $\mathcal{O}(h)$, consistent with the linearised approximation.

2. **Proper distance for an arbitrary arm direction.**

Solution outline.

Key idea. The proper distance between two events separated by coordinate interval ℓ^i is

$$ds^2 = g_{ij}\,\ell^i\ell^j = (\eta_{ij} + h_{ij}^{\mathrm{TT}})\,\ell^i\ell^j.$$

Expanding to first order in h and normalising by $L = |\boldsymbol{\ell}|$ gives the fractional change.

Step 1. Write $\ell^i = L\hat{n}^i$ with $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Expand $ds \simeq L[1 + \frac{1}{2}h_{ij}^{\mathrm{TT}}\hat{n}^i\hat{n}^j]$. The fractional change is $\Delta s/L = \frac{1}{2}h_{ij}^{\mathrm{TT}}\hat{n}^i\hat{n}^j$.

Step 2. Insert the TT metric for a wave along $\hat{z}$:

$$h_{ij}^{\mathrm{TT}}\hat{n}^i\hat{n}^j = h_+\,(\hat{n}_x^2 - \hat{n}_y^2) + 2h_\times\,\hat{n}_x\hat{n}_y.$$

Substituting $\hat{n}_x = \sin\theta\cos\phi$, $\hat{n}_y = \sin\theta\sin\phi$ gives

$$\frac{\Delta s}{L} = \tfrac{1}{2}\big[h_+\,\sin^2\theta\cos 2\phi + h_\times\,\sin^2\theta\sin 2\phi\big].$$

Step 3. For a horizontal arm ($\theta = \pi/2$) and a zenith wave ($\hat{k} = \hat{z}$), the factor $\sin^2\theta = 1$ and the response reduces to $\frac{1}{2}(h_+\cos 2\phi + h_\times\sin 2\phi)$, which is quadrupolar in the azimuth ϕ. The detector is blind along the bisectors of the arms ($\phi = \pi/4 + n\pi/2$) and maximally sensitive along the arms.

Pitfall. The $\cos\theta$ component of $\hat{n}$ does not contribute because the $\hat{z}$-row and $\hat{z}$-column of h_{ij}^{TT} are zero for a wave propagating along $\hat{z}$ (transversality condition).

3. **Sinc factor: one-way vs round-trip.**

Solution outline.

Key idea. The phase accumulated by a photon along one arm in the presence of a monochromatic GW $h(t) = h_0\cos\omega_{\mathrm{gw}}t$ is an integral of the metric along the null geodesic. The round-trip phase is the sum of the outgoing and returning integrals evaluated at shifted times.

Step 1: one-way result. A photon emitted at $t = 0$ reaches the far mirror at $t = L/c$. The phase delay picked up on the outward journey is

$$\Delta\varphi_{\mathrm{one}} = k_L\int_0^{L/c}\frac{h(t')}{2}\,c\,dt' = \frac{k_L L}{2}\,h_0\,\mathrm{sinc}\!\left(\frac{\omega_{\mathrm{gw}}L}{2c}\right)\cos\!\left(\frac{\omega_{\mathrm{gw}}L}{2c}\right). \tag{1.1}$$

The argument of sinc is $\omega_{\mathrm{gw}}L/(2c)$.

Step 2: adding the return leg. The return photon traverses the arm from $t = L/c$ to $t = 2L/c$. Its phase delay has the same amplitude but the integration window is shifted. Adding both legs carefully (the return integral is evaluated over a shifted interval), the combined phase is

$$\Delta\varphi_{\mathrm{rt}} = k_L L\,h_0\,\mathrm{sinc}\!\left(\frac{\omega_{\mathrm{gw}}L}{c}\right)\cos\!\left(\omega_{\mathrm{gw}}t - \frac{\omega_{\mathrm{gw}}L}{c}\right). \tag{1.2}$$

The sinc argument has doubled to $\omega_{\mathrm{gw}}L/c$.

Step 3: long-wavelength limit. For $\omega_{\mathrm{gw}}L/c \ll 1$, sinc $\to 1$, so $\Delta t_{\mathrm{rt}} \simeq h\,L/c$ and $\Delta t_{\mathrm{one}} \simeq h\,L/(2c)$, as required.

Note. The doubling of the sinc argument from one-way to round-trip is not a trivial factor of 2. It arises because the return photon samples a different portion of the GW cycle; the two contributions add coherently only in the long-wavelength limit.

4. **Antenna pattern at a specific configuration.**

Solution outline.

Setup. For a detector at the equator with arms along east ($\hat{x}$) and north ($\hat{y}$), the detector tensor is $D^{ij} = \frac{1}{2}(\hat{x}^i\hat{x}^j - \hat{y}^i\hat{y}^j)$. The antenna pattern functions are $F_+ = D^{ij}e^+_{ij}$ and $F_\times = D^{ij}e^\times_{ij}$.

Part (a): source at zenith, $\psi = 0$. $\theta = 0$, $\phi = 0$ (or any ϕ, since the source is overhead). With the standard formulas:

$$F_+ = \cos 2\psi, \quad F_\times = \sin 2\psi, \quad \Rightarrow \quad F_+^2 + F_\times^2 = 1 \quad \forall\, \psi.$$

At $\psi = 0$: $F_+ = 1$, $F_\times = 0$. At $\psi = \pi/4$: $F_+ = 0$, $F_\times = 1$. The sum is 1 in both cases, confirming independence of ψ.

Part (b): source at the horizon, due east. $\theta = \pi/2$ (on the horizon), $\phi = 0$ (due east, along one arm). Substituting:

$$F_+ = \tfrac{1}{2}(1 + \cos^2(\pi/2))\cos 0\, \cos 2\psi - \cos(\pi/2)\sin 0\, \sin 2\psi = \tfrac{1}{2}\cos 2\psi.$$

$$F_\times = \tfrac{1}{2}\cos 2\psi\, \sin 2\psi + 0 \cdot \ldots$$

For $\psi = 0$: $F_+ = 1/2$, $F_\times = 0$. The response is half the zenith value, because the source in the arm plane couples to the differential arm length at half efficiency. For $\phi = \pi/4$ (source northeast, along the bisector) both F_+ and $F_\times$ vanish: the detector is blind to that direction.

Part (c): sky-averaged response. Average $F_+^2 + F_\times^2$ over ϕ and ψ at fixed θ. The ψ average removes cross terms; the ϕ average is a standard trigonometric integral. The result is

$$\langle F_+^2 + F_\times^2 \rangle_{\phi,\psi} = \tfrac{1}{10}(1 + 6\cos^2\theta + \cos^4\theta).$$

Integrating over the solid angle with weight $\sin\theta\, d\theta$ and dividing by 4π gives the sky average:

$$\langle F_+^2 + F_\times^2 \rangle_{\text{sky}} = \tfrac{2}{5},$$

so the rms directional sensitivity is $\sqrt{2/5} \simeq 0.63$.

Numerical check. A quick consistency check: at $\theta = 0$ (zenith), $(1+6+1)/10 = 0.8$; at $\theta = \pi/2$ (horizon), $(1+0+0)/10 = 0.1$. The zenith direction is 8 times more sensitive than the horizon on average over ϕ.

5. **Ring deformation under a general polarization.**

Solution outline.

Key idea. Use equation $\Delta\xi^i = \frac{1}{2}h_{ij}^{\mathrm{TT}}\xi_0^j$ applied to a ring of unit radius in the x-y plane: $\xi_0 = (\cos\varphi, \sin\varphi, 0)$.

Step 1: deformed position. After the wave acts:

$$\Delta x = \tfrac{h_0}{2}\cos(\omega_{\mathrm{gw}}t)\,\cos\varphi + \tfrac{h_0}{2}\cos(\omega_{\mathrm{gw}}t + \delta)\,\sin\varphi,$$

$$\Delta y = \tfrac{h_0}{2}\cos(\omega_{\mathrm{gw}}t + \delta)\,\cos\varphi - \tfrac{h_0}{2}\cos(\omega_{\mathrm{gw}}t)\,\sin\varphi.$$

Step 2: ellipse and rotation. At any instant, the deformed ring is an ellipse. Its semi-axes and orientation depend on $h_+(t)$ and $h_\times(t)$. For $\delta = 0$ or π (linear polarization), the axes are fixed in space and the shape oscillates in amplitude. For $\delta \neq 0, \pi$ (elliptical polarization), the axes rotate in time.

Step 3: circular case $\delta = \pi/2$, equal amplitudes. Set $h_+(t) = h_0 \cos \omega_{\rm gw} t$ and $h_\times(t) = h_0 \cos(\omega_{\rm gw} t + \pi/2) = -h_0 \sin \omega_{\rm gw} t$. The deformed coordinates become

$$\begin{aligned} x(\varphi, t) &= \cos\varphi + \tfrac{h_0}{2} \cos\varphi \cos\omega_{\rm gw} t - \tfrac{h_0}{2} \sin\varphi \sin\omega_{\rm gw} t = \cos\varphi + \tfrac{h_0}{2} \cos(\varphi + \omega_{\rm gw} t), \\ y(\varphi, t) &= \sin\varphi + \tfrac{h_0}{2} \sin\varphi \cos\omega_{\rm gw} t + \tfrac{h_0}{2} \cos\varphi \sin\omega_{\rm gw} t = \sin\varphi + \tfrac{h_0}{2} \sin(\varphi + \omega_{\rm gw} t). \end{aligned}$$

This is a circle of unit radius plus a circle of radius $h_0/2$ rotating at angular frequency $\omega_{\rm gw}$: the ellipse shape is constant (it is a circle deformed by a fixed amount) but its axes rotate at the GW frequency.

Pitfall. The deformation is still of order $h_0 \ll 1$; the ring does not become circular. What rotates is the orientation of the slightly elliptical perturbation, not the ring itself.

Chapter 2

Block 2 — Experimental Fundamentals

1. **Michelson output with an arm-length offset.**

Solution outline.

Key idea. The output power of a Michelson is $P_{\text{out}} = P_{\text{in}} \sin^2(\Delta\varphi/2)$, where $\Delta\varphi = 4\pi(L_x - L_y)/\lambda_L$ is the differential phase. Park the interferometer at $\Delta\varphi_0$ (set by the static offset δ) and expand for a small GW-induced additional phase $\delta\varphi_{\text{gw}}$.

Step 1: static phase and DC offset. The static arm imbalance $L_x - L_y = 2\delta$ gives $\Delta\varphi_0 = 8\pi\delta/\lambda_L$. For the dark fringe $\Delta\varphi_0 = 0$ ($\delta = 0$). A small offset parks the instrument at $\Delta\varphi_0 \ll 1$.

Step 2: linearised photocurrent. Expanding P_{out} to first order in both δ and h:

$$P_{\text{out}} \simeq P_{\text{in}} \sin^2(\Delta\varphi_0/2) + P_{\text{in}} \sin(\Delta\varphi_0) \cdot \frac{\delta\varphi_{\text{gw}}}{2}.$$

The GW-induced phase is $\delta\varphi_{\text{gw}} = 4\pi L h(t)/\lambda_L$. The signal photocurrent (AC term) is linear in h and proportional to $\sin\Delta\varphi_0 \approx \Delta\varphi_0 \propto \delta$.

Step 3: optimal offset. Two noise contributions compete:

- Shot noise on the photocurrent $\propto \sqrt{P_{\text{out}}} \propto \sqrt{\delta}$, so the shot-noise contribution to h_{shot} scales as $(\delta)^{-1/2}$: it improves with larger δ.
- Laser intensity noise (RIN) produces a spurious power fluctuation $\delta P = \text{RIN} \times P_{\text{DC}} \propto \delta^2$ (since $P_{\text{DC}} \propto \delta^2$ near the dark fringe), which referred to h scales as δ.

Equating the two noise contributions:

$$\frac{\sqrt{\hbar\omega_L c}}{2L\sqrt{P_{\text{in}}}} \cdot \frac{1}{\sqrt{\delta}} \sim \frac{\lambda_L}{4\pi L} \text{RIN} \cdot \delta,$$

gives an optimal offset $\delta_{\text{opt}} \sim \lambda_L/(4\pi) \cdot (\text{RIN}\sqrt{P_{\text{in}}/\hbar\omega_L c})^{-2/3}$.

Numerical check. For $P_{\text{in}} = 10$ W, $\lambda_L = 1064$ nm, $\text{RIN} = 10^{-8}\ \text{Hz}^{-1/2}$: δ_{opt} is of order a few picometres, corresponding to a DC power at the output port of order μW. This is the DC readout scheme used in Advanced Virgo and Advanced LIGO.

2. **Fabry–Perot cavity pole and finesse.**

Solution outline.

Key idea. Solve the round-trip self-consistency condition for the circulating field and extract the transfer function as a function of frequency.

Step 1: circulating field. From $E_{\text{circ}} = t_1 E_{\text{in}} + r_1 r_2 e^{2ik_L L} E_{\text{circ}}$, solve:

$$E_{\text{circ}} = \frac{t_1\, E_{\text{in}}}{1 - r_1 r_2\, e^{2ik_L L}}.$$

On resonance ($e^{2ik_L L} = 1$), the circulating power is $|t_1|^2/(1 - r_1 r_2)^2$ times the input power.

Step 2: finesse. The finesse is the ratio of the free spectral range to the linewidth. Taking the magnitude of the denominator,

$$|1 - r_1 r_2 e^{i\phi}|^2 = (1 - r_1 r_2)^2 + 4 r_1 r_2 \sin^2(\phi/2).$$

The FWHM of the resonance is the ϕ range over which the denominator doubles. Setting $4 r_1 r_2 \sin^2(\phi_{1/2}/2) = (1 - r_1 r_2)^2$ gives $\sin(\phi_{1/2}/2) \approx (1 - r_1 r_2)/(2\sqrt{r_1 r_2})$ for high reflectivity. The finesse is

$$\mathcal{F} = \frac{\pi}{\phi_{1/2}} = \frac{\pi\sqrt{r_1 r_2}}{1 - r_1 r_2}.$$

Step 3: cavity pole. Near resonance, write $\phi = 4\pi f L/c$ for a small frequency offset f from resonance. The cavity transfer function has a pole at

$$f_{\text{pole}} = \frac{c}{4\mathcal{F}L}.$$

Numerical check for Advanced Virgo arm. $r_1^2 = 0.986 \Rightarrow r_1 = 0.9930$; $r_2 = 0.99999$.

$$\mathcal{F} = \frac{\pi \times 0.9930 \times 0.99999}{1 - 0.9930 \times 0.99999} \approx \frac{\pi \times 0.9930}{0.00702} \approx 444.$$

$$f_{\text{pole}} = \frac{3 \times 10^8}{4 \times 444 \times 3 \times 10^3} \approx 56 \text{ Hz}.$$

These match the published Advanced Virgo arm parameters ($\mathcal{F} \approx 443$, $f_{\text{pole}} \approx 56$ Hz).

3. **Pendulum thermal noise: viscous vs structural damping.**

Solution outline.

Key idea. The Fluctuation–Dissipation Theorem (FDT) relates the displacement power spectrum to the imaginary part of the mechanical susceptibility. The susceptibility changes depending on whether the loss is viscous (velocity-proportional damping) or structural (frequency-independent loss angle).

Step 1: viscous damping. The susceptibility is $\alpha(\omega) = 1/[m(\omega_0^2 - \omega^2 + i\omega\omega_0/Q)]$. In the low-frequency limit $\omega \ll \omega_0$:

$$S_x^{\text{visc}}(\omega) = \frac{4k_B T}{m} \frac{\omega_0/Q}{(\omega_0^2 - \omega^2)^2 + (\omega\omega_0/Q)^2} \xrightarrow{\omega \ll \omega_0} \frac{4k_B T}{m\omega_0^3 Q} \quad [\text{flat in } \omega].$$

The spectrum is frequency-independent below the resonance: thermal noise does not fall off toward lower frequencies in this model.

Step 2: structural damping. Replace the viscous loss $i\omega\omega_0/Q$ with the structural loss $i\omega_0^2\phi$ (loss angle ϕ, assumed frequency-independent). The susceptibility is $\alpha(\omega) = 1/[\kappa(1+i\phi) - m\omega^2]$ where $\kappa = m\omega_0^2$. For $\omega \ll \omega_0$:

$$\mathrm{Im}[\alpha(\omega)] \approx -\frac{\kappa\phi}{(\kappa)^2} = -\frac{\phi}{m\omega_0^2} \quad \text{(frequency-independent)}.$$

The FDT gives

$$S_x^{\text{struct}}(\omega) = \frac{4k_BT}{\omega}\,|\mathrm{Im}[\alpha(\omega)]| \;\propto\; \frac{1}{\omega}\,.$$

Below the resonance, the structural-damping noise spectrum falls as $1/\omega$ (i.e., as $1/f$ in terms of strain spectral density), while the viscous-damping spectrum is flat. For $\phi \ll 1$, the structural-damping level at the resonance frequency is the same as the viscous case (since $\phi = 1/Q$), but at low frequencies structural damping gives less noise because its $1/\omega$ fall-off runs away from the flat viscous floor.

> **Note.** This is the physical motivation behind the move to fused-silica monolithic suspensions (in Advanced Virgo and Advanced LIGO) and silicon ribbon suspensions (proposed for cryogenic third-generation detectors): both achieve small, nearly frequency-independent ϕ, suppressing thermal noise in the detection band. See Saulson, *Fundamentals of Interferometric Gravitational-Wave Detectors*, Ch. 7, for the full derivation.

4. **Optimal laser power at the SQL.**

Solution outline.

Key idea. Shot noise and radiation-pressure noise have opposite dependences on circulating power P. Minimising their sum over P at fixed frequency gives the Standard Quantum Limit.

Step 1: the two noise contributions.

$$S_h^{\text{shot}}(f) = \frac{\hbar c \lambda_L}{4\pi^2 L^2 P},$$
$$S_h^{\text{rp}}(f) = \frac{16\pi^2 P}{\hbar c\lambda_L\, m^2\omega^4 L^2}\cdot\frac{\hbar^2 c^2}{1} = \frac{16\hbar P}{m^2\omega^4 L^2 c}.$$

(Using $\omega = 2\pi f$.) The first grows as P^{-1}; the second grows as P.

Step 2: minimise over P. Set $dS_h^{\text{total}}/dP = 0$:

$$P_{\text{opt}}(f) = \frac{m\omega^2 c\lambda_L}{4\pi\sqrt{4}}\cdot\ldots$$

The minimisation gives

$$S_h^{\text{shot}} + S_h^{\text{rp}}\big|_{P_{\text{opt}}} = 2\sqrt{S_h^{\text{shot}}\cdot S_h^{\text{rp}}} = \frac{8\hbar}{mL^2\omega^2} \equiv S_h^{\text{SQL}}(f).$$

Numerical check. For $m = 40$ kg, $L = 4$ km, $f = 100$ Hz, $\lambda_L = 1064$ nm:

$$P_{\text{opt}} = \frac{m\omega^2 L^2}{8\hbar/(\hbar c\lambda_L/(4\pi^2))} \approx 800\ \text{kW}.$$

Advanced LIGO has a circulating power of about 200–400 kW in the arms, a factor of two to four below P_{opt} at 100 Hz. Operating below P_{opt} means the instrument is shot-noise dominated in the bucket. Squeezing allows one to break the SQL without increasing power: by injecting frequency-dependent squeezed vacuum, both shot noise and radiation-pressure noise can be reduced simultaneously at different frequencies.

Pitfall. P_{opt} depends strongly on frequency: at 10 Hz it is 100 times larger than at 100 Hz. The SQL is therefore not a single number; it is a frequency-dependent envelope. Frequency-dependent squeezing is needed to approach it uniformly across the band.

5. **Triangulation from timing.**

Solution outline.

Key idea. The time delay Δt between two detectors constrains the source direction to a locus where the path-length difference to the two sites equals $c\,\Delta t$. In 3D this locus is a cone; projected onto the celestial sphere it is a ring.

Step 1: geometry. Let $\hat{\Omega}$ be the unit vector toward the source and $\vec{b} = \vec{r}_2 - \vec{r}_1$ the baseline vector. The time delay is

$$\Delta t = \frac{\vec{b} \cdot \hat{\Omega}}{c}.$$

This fixes $\cos\theta = c\,\Delta t/|\vec{b}|$, where θ is the angle between $\hat{\Omega}$ and $\hat{b}$. For each value of Δt the source lies on a ring (a small circle on the sphere) with half-opening angle θ around the baseline direction.

Step 2: ring width. Differentiating, $d(\cos\theta) = c\,d(\Delta t)/|\vec{b}|$, so the angular uncertainty of the ring is

$$\delta\theta \approx \frac{c\,\sigma_{\Delta t}}{|\vec{b}|\,\sin\theta}.$$

For the LIGO-Hanford/LIGO-Livingston baseline ($|\vec{b}| \simeq 3000$ km), $c\,\sigma_{\Delta t} \simeq 3 \times 10^8 \times 10^{-3} = 3 \times 10^5$ m $=$ 300 km. The ring width is $\delta\theta \approx 300/3000\,\text{rad} \approx 6^\circ$, i.e., a band $\sim 12^\circ$ wide around the sky.

Step 3: role of the third detector. Two detectors produce one ring. A third detector at a different location produces a second ring; the source lies at their intersection. For LIGO + Virgo the two rings intersect in one or two patches, reducing the localisation area from thousands of square degrees to tens (or a few hundred for low-SNR events). Adding KAGRA further breaks the degeneracy. The median 90% credible area during O4 for a three-detector network is of order 10–100 deg^2, compared with ~ 1000 deg^2 for two detectors alone.

Numerical check. The GW170817 BNS event was localised to 28 deg^2 (90% CL) by the LIGO–Virgo network, enabling the electromagnetic follow-up that led to the identification of the host galaxy NGC 4993 within hours.