GW theoretical aspects and principia

Lecture 2: gravitational waves on curved backgrounds

SVT decomposition, cosmological propagation, and ringdown

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When flat spacetime is no longer enough

Flat-space description. Over one wavelength the background looks almost flat

$$\lambda_{GW} \ll L_R \sim |\bar{R}_{\mu\nu\rho\sigma}|^{-1/2}$$

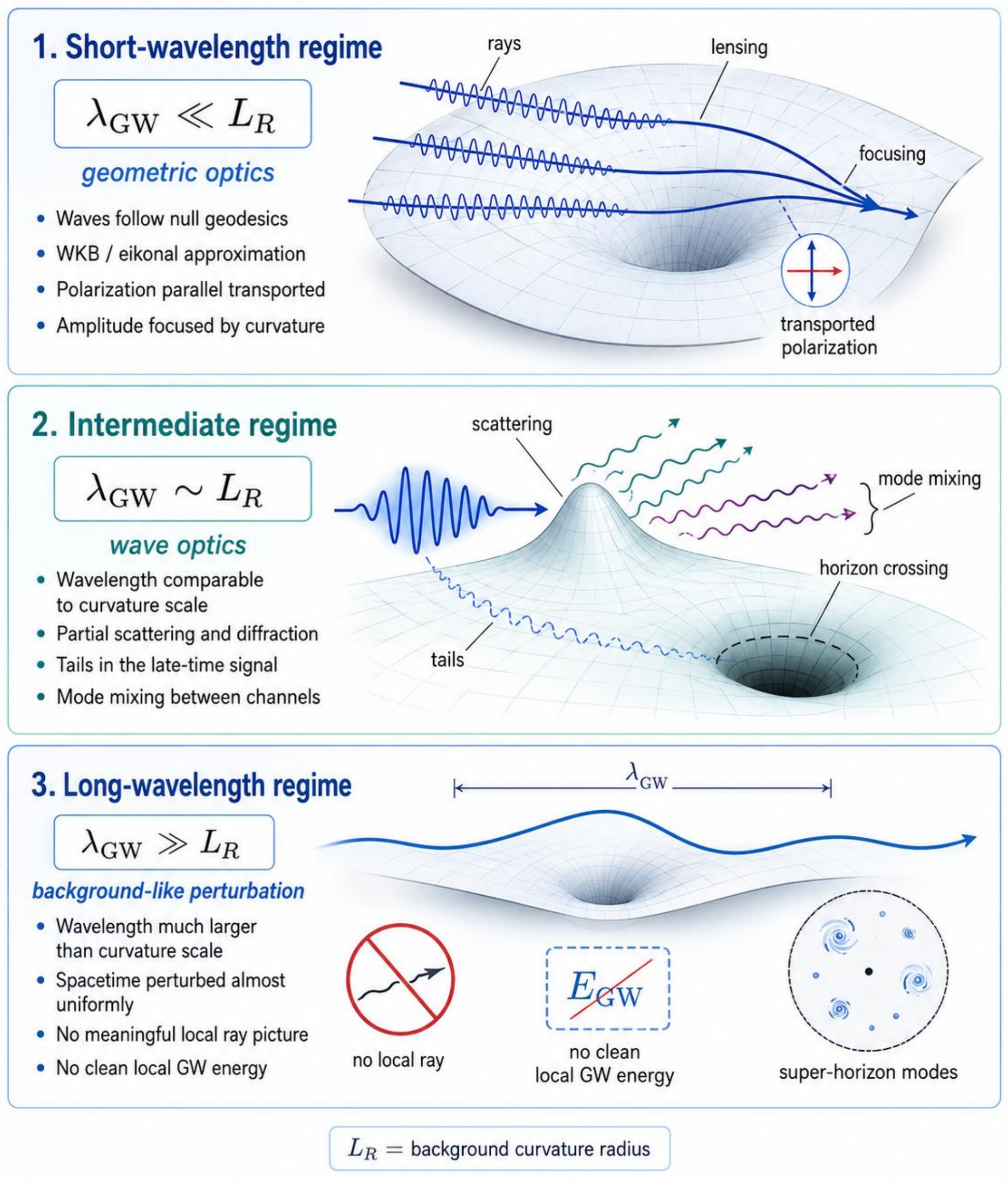
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\Box h_{ij}^{TT} = 0$$

Curved background description. Perturbation of a non-trivial geometry: the background affects propagation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Minkowski is the local limit.
Curved backgrounds are needed when the wave samples expansion or strong curvature over an extended region



Gravitational lensing of GW

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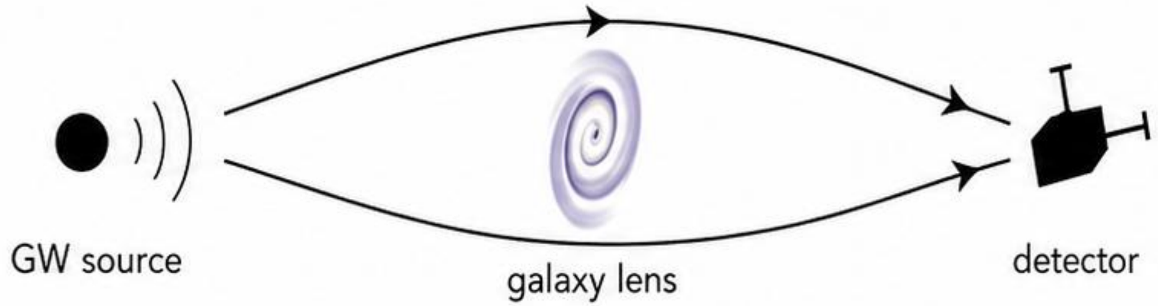
Propagation through an inhomogeneous Universe can lens gravitational-wave signals, much as for light.

Basic picture and examples



Einstein ring

Multiple images

Galaxy-cluster lensing

What lensing does

Analogous to gravitational lensing of light, but observed in the gravitational-wave signal.

Observable effects

- magnification / demagnification
- multiple “images” with time delays
- frequency-dependent waveform distortions

Potential applications

- tests of fundamental physics
- localization of merging black holes
- precision cosmology
- study of microlens populations

Lensing encodes information on both the source and the intervening matter distribution.

Minkowski, FRW, and black holes as three laboratories

Same small perturbation. Different background symmetries. Different physical questions.

common split $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad |h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$ background sets the symmetry, decomposition, and wave equation

Minkowski

Poincaré symmetry

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$$

local generation and detector response

plane waves · TT gauge · quadrupole input

FRW

homogeneous + isotropic

$$ds^2 = a^2(\eta)(-d\eta^2 + \gamma_{ij}dx^i dx^j)$$

cosmological propagation

redshift · amplitude damping · luminosity distance

Black holes

spherical / axial symmetry

$$\bar{g}_{\mu\nu} = g_{\mu\nu}^{\mathrm{Schw/Kerr}}$$

strong-field scattering

horizons · effective potentials · QNMs

local grammar

cosmological syntax

strong-field spectrum

Takeaway: the wave is always a metric perturbation, but the background decides what “propagation” means.

Gauge freedom beyond flat spacetime

$$h_{\mu\nu} \to h_{\mu\nu} - \bar\nabla_\mu \xi_\nu - \bar\nabla_\nu \xi_\mu$$

equivalently $\delta_\xi h_{\mu\nu} = -2\,\bar\nabla_{(\mu}\xi_{\nu)}$

flat-space rule $\partial_\mu \to \bar\nabla_\mu$

what changes

partial derivatives become background-covariant

trace: $h \to h - 2\bar\nabla_\mu \xi^\mu$

the background connection enters the transformation

Lorenz-like gauge

$$\bar\nabla_\mu \bar h^\mu{}_\nu = 0$$

$$\bar h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar g_{\mu\nu} h$$

residual freedom: $\bar\Box \xi_\nu + \bar R_{\nu\rho}\xi^\rho = 0$

physical lesson

do not copy the Minkowski TT gauge blindly

gauge conditions must be adapted to the background

use SVT in FRW, master variables near black holes

Flat space: $\nabla- \to \partial$, $\bar R_{\mu\nu\rho\sigma} = 0$,
global TT gauge for plane waves

Curved space: $\bar\nabla \neq \partial$, $\bar R_{\mu\nu\rho\sigma} \neq 0$,
observables must be gauge-invariant or fully gauge-fixed

Gauge freedom is not lost on a curved background. It becomes more geometric: the background enters through $\bar\nabla_\mu$ and through the residual gauge equation.

Background curvature enters the perturbation equations explicitly.

Background curvature enters the perturbation equations

The flat wave equation becomes background-covariant and gains curvature coupling

$$\delta R_{\mu\nu} = -\tfrac{1}{2}\bar{\nabla}^\rho\bar{\nabla}_\rho h_{\mu\nu} - \tfrac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}_\rho\bar{\nabla}_{(\mu}h_{\nu)}{}^\rho + \bar{R}_{\mu\rho\nu\sigma}h^{\rho\sigma} - \bar{R}_{\rho(\mu}h_{\nu)}{}^\rho$$

overbars : $\bar{g}_{\mu\nu}$, $\bar{\nabla}_\mu$, $\bar{R}_{\mu\nu\rho\sigma}$ are background quantities

curved wave operator

$$\partial^\rho\partial_\rho \longrightarrow \bar{\nabla}^\rho\bar{\nabla}_\rho = \bar{g}^{\rho\sigma}\bar{\nabla}_\rho\bar{\nabla}_\sigma$$

the propagation operator now contains the background metric

curvature coupling

$$\bar{R}_{\mu\rho\nu\sigma}h^{\rho\sigma}, \qquad \bar{R}_{\rho(\mu}h_{\nu)}{}^\rho$$

these terms vanish in Minkowski but not on a curved background

physical effects

FRW: redshift and Hubble damping
black holes: scattering and tails
ringdown: effective potentials

Flat space: $\bar{R}_{\mu\nu\rho\sigma} = 0, \quad [\partial_\mu, \partial_\nu] = 0$
simple wave equation for the perturbation

Curved space: $[\bar{\nabla}_\mu, \bar{\nabla}_\nu]V^\rho = \bar{R}^\rho{}_{\sigma\mu\nu}V^\sigma$
non-commuting derivatives create curvature terms

Background curvature **changes the operator that propagates the perturbation** and **adds curvature-cooupling terms**.
This is the mathematical origin of cosmological damping and black-hole scattering

Symmetry and decoupling of perturbations

Key idea: decompose the perturbation field according to the background symmetries.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu$$

Coordinate components are gauge dependent.

metric perturbation $h_{\mu\nu}$

- **S — Scalar sector**: background scalars (e.g., density, curvature)
- **V — Vector sector**: divergence-free vector modes
- **T — Tensor sector**: transverse-traceless tensor modes

GW sector in FRW

Background-adapted decompositions

- Minkowski: Fourier modes + TT polarizations
- FRW: scalar-vector-tensor (SVT) on spatial slices
- Schwarzschild / Kerr: spherical or spheroidal harmonics

FRW / SVT (on spatial slices with metric γ_{ij})

$$V_i = D_i V + V_i^T, \quad D^i V_i^T = 0$$

$$h_{\mu\nu} = h_{\mu\nu}^S + h_{\mu\nu}^V + h_{\mu\nu}^T$$

$$D^i H_{ij} = 0, \quad \gamma^{ij} H_{ij} = 0$$

D_i: covariant derivative on spatial slices;
T: transverse (divergence-free);
H_{ij}: transverse-traceless tensor.

At linear order the scalar, vector, and tensor sectors decouple, because the linearized Einstein operator respects the background isometries.

Therefore, the physical gravitational-wave degrees of freedom are isolated in the transverse-traceless tensor sector.

The spatial background: maximally symmetric 3-geometry

SVT decomposition is defined on the spatial slices of FRW. Their symmetry is the reason scalar, vector, and tensor sectors can be organized separately.

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}(\vec{x})\,dx^i\,dx^j$$

γ_{ij} : metric of a 3-space of constant curvature

For spatial flatness: $\gamma_{ij} = \delta_{ij}$

FRW background

time direction | spatial slices

Flat $(k = 0)$

Closed $(k = +1)$

Open $(k = -1)$

Why maximally symmetric?

- homogeneous and isotropic spatial slices
- 6 Killing vectors on each slice
- 3 rotations + 3 translations (in the flat case)
- these symmetries provide the natural basis for perturbations

Spatial operators

D_i : covariant derivative compatible with γ_{ij}

$D^2 = \gamma^{ij} D_i D_j$

All SVT variables are defined with respect to γ_{ij}

Key consequence

Background symmetry → irreducible decomposition → block-diagonal linear equations

For FRW: homogeneity + isotropy → scalar / vector / tensor sectors

This spatial geometry is the stage on which the scalar–vector–tensor decomposition is performed.

Scalars, vectors, and tensors on a 3-space

SVT is a geometrical decomposition of spatial fields. It separates longitudinal, transverse, and transverse-traceless pieces.

Vector field

$$V_i = \partial_i V^S + V_i^V, \qquad \partial^i V_i^V = 0$$

$$3 = 1 + 2$$

scalar potential + transverse vector

Symmetric spatial tensor

$$X_{ij} = \frac{1}{3}\delta_{ij}X + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)X^S + 2\partial_{(i}X_{j)}^V + X_{ij}^T$$

$$\partial^i X_i^V = 0, \qquad \partial^i X_{ij}^T = 0, \qquad \delta^{ij}X_{ij}^T = 0$$

trace scalar	longitudinal scalar	transverse vector	TT tensor
1	1	2	2

$$6 = 1 + 1 + 2 + 2$$

Fourier-space picture

For each non-zero $\vec{k}$:

longitudinal: parallel to $\vec{k}$

transverse: orthogonal to $\vec{k}$

$$P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$$

$$X_{ij}^{TT} = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)X_{kl}$$

What S, V, T mean

S, V, T refer to spatial rotations, not spacetime tensor rank.
Example: $\partial_i V^S$ has an index, but belongs to the scalar sector.

Tensor sector

For $\vec{k} \parallel \hat{z}$:

$$X_{ij}^T = \begin{pmatrix} X_+ & X_\times & 0 \\ X_\times & -X_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

two TT functions $\longrightarrow$ two GW polarizations

The SVT split is complete and kinematical: no component is lost, and no gauge has been chosen.
Applied to $h_{\mu\nu}$, the TT tensor piece is the gravitational-wave sector.

The SVT decomposition of metric perturbations

On a spatially flat FRW background, the 10 components of $h_{\mu\nu}$ split into scalar, vector, and tensor sectors.

Starting point

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$d\bar{s}^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$$

background = spatially flat FRW

Metric perturbation decomposition

$$h_{00} = -2A$$

$$h_{0i} = a(\partial_i B + S_i), \qquad \partial_i S_i = 0$$

$$h_{ij} = a^2\left[2\delta_{ij}\psi + 2\partial_i\partial_j E + \partial_{(i}F_{j)} + H_{ij}\right]$$

$$\partial_i F_i = 0, \qquad \partial_i H_{ij} = 0, \qquad \delta_{ij} H_{ij} = 0$$

Interpretation

- **Scalars** describe the lapse/shift and spatial scalar distortions.
- **Vectors** are transverse and represent rotational-type perturbations.
- H_{ij} is transverse-traceless (TT).

Scalar sector

$A,\; B,\; \psi,\; E$

4 scalar functions

Vector sector

$S_i\;(2) + F_i\;(2)$

4 vector components

Tensor sector

H_{ij}

2 tensor degrees of freedom

$$4 + 4 + 2 = 10$$

matches the 10 components of a symmetric metric perturbation $h_{\mu\nu}$.

No degree of freedom is lost: SVT is a complete kinematical split.

In cosmology, the tensor piece H_{ij} is the gravitational-wave sector.

Scalar perturbations

Scalar modes are essential for cosmology, but in pure GR they are not independent gravitational waves.

Scalar metric sector

$$ds_S^2 = -(1+2A)\,dt^2 + 2a\,\partial_i B\,dt\,dx^i + a^2[(1+2\psi)\delta_{ij} + 2\partial_i\partial_j E]\,dx^i\,dx^j$$

Four scalar functions

- **A** : lapse / time-time perturbation
- **B** : scalar part of the shift
- **Ψ** : spatial curvature perturbation
- **E** : longitudinal scalar shear

Newtonian gauge

$$B = E = 0$$

Two scalar potentials: **Φ , Ψ**

Sub-horizon constraint: $\nabla^2\Psi \simeq 4\pi G\, a^2\, \delta\rho$

Matter sources

$$\delta\rho, \quad \delta p, \quad v, \quad \pi$$

Density, pressure, scalar velocity potential, and anisotropic stress source scalar metric perturbations.

Physical message

Scalar modes control density growth, large-scale structure, and most CMB temperature anisotropies.

They are not a new spin-0 gravitational wave in pure GR; the propagating GW sector is the transverse-traceless tensor h_{ij}.

Vector perturbations

Vector modes describe rotational perturbations of the metric. In standard cosmology they usually decay in the absence of sustained sources.

Vector metric sector

$$ds_V^2 = -dt^2 + 2aS_i\, dt\, dx^i + a^2 (\partial_i F_j + \partial_j F_i)\, dx^i dx^j$$

with $\partial_i S^i = 0, \quad \partial_i F^i = 0$

Transverse vectors

- S_i : vector part of the shift
- F_i : vector part of the spatial metric
- each has 2 independent components

Gauge-invariant combination

$$\Sigma_i = S_i + a\dot{F}_i$$

This combination is unchanged by vector gauge transformations.

Sourcing

Vector modes are sourced by vorticity or anisotropic stress. Perfect fluids without vorticity do not sustain them.

Physical message

In GR and in the absence of sources, vector perturbations typically decay with the expansion. They are therefore usually subdominant in cosmology.

Tensor perturbations

Tensor modes are the transverse-traceless perturbations of the spatial metric. They are the propagating gravitational waves.

Tensor metric sector

$$ds_T^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j$$

$$\partial^i h_{ij} = 0, \qquad h^i_i = 0$$

Transverse-traceless field

- symmetric spatial tensor h_{ij}
- two independent components
- gauge-invariant at linear order

Two polarizations

$$h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij}$$

plus and cross modes

Evolution in Fourier space

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = 16\pi G a^2 \Pi^{TT}_{ij}$$

In vacuum: damped waves on the expanding background.

Physical message

Tensor modes are the genuine GW sector. They carry the radiative degrees of freedom and can be sourced by transverse-traceless anisotropic stress.

Decoupling at linear order

On an exactly homogeneous and isotropic background, symmetry plus linearity keeps scalar, vector, and tensor sectors independent.

SVT split

$$h_{\mu\nu} = {h_{\mu\nu}}^{S} + {h_{\mu\nu}}^{V} + {h_{\mu\nu}}^{T}$$

scalar vector tensor

$$\text{spin } 0 \oplus \text{spin } 1 \oplus \text{spin } 2$$

Symmetry argument

$$\mathcal{L}\,\mathcal{R} = \mathcal{R}\,\mathcal{L}$$

The linearized Einstein operator is built from the FRW background.
It respects rotations and cannot change spin.

Block diagonal form

$$\mathcal{L} = \mathcal{L}_S \oplus \mathcal{L}_V \oplus \mathcal{L}_T$$

No off-diagonal mixing between S, V, and T sectors at first order.

Independent equations

$$\delta G^{S}_{\mu\nu} = 8\pi G\,\delta T^{S}_{\mu\nu}$$
$$\delta G^{V}_{\mu\nu} = 8\pi G\,\delta T^{V}_{\mu\nu}$$
$$\delta G^{T}_{\mu\nu} = 8\pi G\,\delta T^{T}_{\mu\nu}$$

Each equation sees only the source with the same rotational type.

symmetry + linearity ⇒ decoupling

Physical message

S × S → T only beyond linear theory

Scalar perturbations do not source tensor waves at linear order.
The tensor field H_{ij} can be studied as an independent gravitational-wave sector.
Mode mixing appears only beyond linear theory, or if the background symmetry is reduced.

Gauge transformations in the SVT language

Gauge transformations reshuffle coordinate-dependent pieces, but they do not mix scalar, vector, and tensor sectors.

Gauge generator

$$\xi^\mu = (\xi^0, \partial_i \xi^L + \xi_i^V),$$
$$\partial^i \xi_i^V = 0.$$

two scalar functions + one transverse vector

Scalar sector

$$A \to A - \dot{\xi}^0;$$
$$\psi \to \psi - H\xi^0;$$
$$B \to B - \frac{\xi^0}{a} + a\dot{\xi}^L;$$
$$E \to E - \xi^L.$$

Vector sector

S_i and F_i shift only under ξ_i^V.

no scalar-vector mixing.

Tensor sector

$$H_{ij} \to H_{ij}.$$

transverse-traceless tensor is gauge invariant at linear order.

Physical meaning

Gauge choice changes the slicing and threading, not the tensor gravitational wave. H_{ij} is already a physical variable on an FRW background.

Gauge-invariant variables: what survives coordinates?

Coordinate choices move A, B, ψ, E. The physical scalar information is in invariant combinations.

Gauge-dependent scalars

$$A,\ B,\ \psi,\ E$$

lapse, shift, spatial curvature, shear

Bardeen potentials

$$\Phi = A + \frac{1}{a}\frac{d}{dt}\left[a\left(B - a\dot{E}\right)\right]$$

$$\Psi = -\psi - H\left(B - a\dot{E}\right)$$

Longitudinal gauge

$$B = E = 0$$

$$\Phi = A, \qquad \Psi = -\psi$$

Scalar metric in Newtonian form

$$ds^2 = -(1 + 2\Phi)\,dt^2$$

$$+\,a^2(t)(1 - 2\Psi)\,\delta_{ij}\,dx^i dx^j$$

Sub-horizon constraint:

$$\nabla^2\Psi \simeq 4\pi G a^2 \delta\rho$$

Scalar potentials are physical, but not spin-2 waves.

Vector / tensor

$$\Sigma_i = S_i - a\dot{F}_i$$

vector gauge invariant

$$\dot{\Sigma}_i + 2H\Sigma_i = 0$$

decays without sources

$$H_{ij} \to H_{ij}$$

TT tensor is gauge invariant.

Physical message

Gauge-invariant means:

not a coordinate artifact.

It does not automatically mean:

propagating radiation.

Φ, Ψ are physical scalar potentials.
Only H_{ij} propagates in pure GR.

Which SVT pieces propagate in GR?

Gauge invariant does not automatically mean propagating: in linear vacuum GR on an FRW background, only the tensor sector carries radiative degrees of freedom.

Scalar sector

- Gauge-invariant variables: Φ, Ψ
- Physical scalar potentials
- In pure GR: constraint equations only
- No second time derivatives in the vacuum scalar sector
- Therefore: no propagating scalar gravitational waves

Poisson-type / constraint-like, not wave-like

Vector sector

- Gauge-invariant variable: Σ_i
- Transverse vector perturbation
- In pure GR vacuum: no source-driven vector radiation
- Evolution is decaying, schematically:

$$\dot{\Sigma}_i + 2H\Sigma_i = 0$$

- Therefore: no propagating vector gravitational waves

$\Sigma_i \propto a^{-2}$ (decays)

Tensor sector

- Gauge-invariant variable: H_{ij}
- Transverse-traceless tensor
- This is the genuine gravitational-wave sector
- Satisfies a hyperbolic wave equation

$$\ddot{H}_{ij} + 3H\dot{H}_{ij} - \frac{1}{a^2}\nabla^2 H_{ij} = 0$$

- Therefore: propagates with 2 polarizations $(+, \times)$

Degree counting: 10 components of $h_{\mu\nu}$ − 4 gauge freedoms − 4 constraints = 2 physical propagating degrees of freedom.

In linear GR on FRW: scalar = constrained, vector = constrained/decaying, tensor = propagating.

Tensor modes as gravitational waves

On an FRW background, the transverse-traceless tensor H_{ij} is the curved-space generalization of h_{ij}^{TT}.
It carries the two gravitational-wave polarizations and propagates on the expanding universe.

Metric and TT conditions

- Pure tensor sector:

$$ds^2 = -dt^2 + a^2(t)\,[\delta_{ij} + H_{ij}(t, \mathbf{x})]\,dx^i dx^j$$

- Transverse condition:

$$\partial_i H_{ij} = 0$$

- Traceless condition:

$$\delta_{ij} H_{ij} = 0$$

- Component count:

$$6 - 3 - 1 = 2$$

Thus $\boldsymbol{H_{ij}}$ contains exactly two independent functions.

Polarizations

- For a Fourier mode with $\boldsymbol{k}$ parallel to z:

$$H_{ij} = \begin{pmatrix} H_+ & H_\times & 0 \\ H_\times & -H_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Two independent tensor amplitudes: $H_+, H_\times$
- Same polarization structure as flat-space TT waves
- Optional compact expansion:

$$H_{ij} = \sum_{\lambda=+,\times} h_\lambda(t, \boldsymbol{k})\, e_{ij}^\lambda\,(\hat{\boldsymbol{k}})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

The FRW expansion changes the evolution, not the number of polarizations.

Propagation and interpretation

- Source-free tensor equation:

$$\ddot{H}_{ij} + 3H\dot{H}_{ij} - \frac{1}{a^2}\nabla^2 H_{ij} = 0$$

- For each polarization:

$$\ddot{h}_\lambda + 3H\dot{h}_\lambda + \frac{k^2}{a^2}h_\lambda = 0$$

- Local sub-horizon limit: if $k/a \gg H$, the wave behaves like a flat-space plane wave
- Global cosmological effect: frequency redshift and amplitude dilution by expansion
- Local detector response:

$$\frac{\Delta L}{L} = \frac{1}{2} H_{ij}\, e^i e^j$$

Locally: ordinary GW physics.
Globally: cosmological propagation.

Why $\boldsymbol{H_{ij}}$ is the gravitational-wave field: gauge invariant • decoupled at linear order • dynamical

H_{ij} on FRW is the direct curved-background analogue of h_{ij}^{TT} in Minkowski spacetime.

Minkowski limit: SVT reduces to TT waves

Set $a(t) = 1$ and $H = 0$: the FRW tensor sector becomes the flat-spacetime transverse-traceless wave of Lecture 1.

FRW tensor sector

$$ds^2 = -dt^2 + a^2(t)\,[\delta_{ij} + h_{ij}^{TT}]dx^i dx^j$$

$$\partial_i h_{ij}^{TT} = 0, \qquad \delta^{ij} h_{ij}^{TT} = 0$$

$$\ddot{h}_{ij}^{TT} + 3H\dot{h}_{ij}^{TT} - \frac{1}{a^2}\nabla^2 h_{ij}^{TT} = 0$$

Two polarizations are encoded in the transverse-traceless tensor field.

Take the flat limit

$$a(t) \to 1$$

$$\dot{a} \to 0, \qquad H \equiv \dot{a}/a \to 0$$

$$\frac{1}{a^2}\nabla^2 \to \nabla^2$$

The expanding background disappears, and the Hubble-friction term vanishes.

Flat-space TT waves

$$ds^2 = -dt^2 + [\delta_{ij} + h_{ij}^{TT}]\,dx^i dx^j$$

$$\partial_i h_{ij}^{TT} = 0, \qquad \delta^{ij} h_{ij}^{TT} = 0$$

$$\ddot{h}_{ij}^{TT} - \nabla^2 h_{ij}^{TT} = 0$$

This is exactly the flat-spacetime transverse-traceless gravitational wave.

Consistency check

FRW tensor mode h_{ij}^{TT} $\longrightarrow$ Minkowski TT mode h_{ij}^{TT}

Same TT constraints, same two polarizations, same wave equation structure. Only the cosmological propagation effects are removed in the Minkowski limit.

FRW spacetime as the cosmological case

Tensor perturbations on an expanding background

Background metric

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$$

$$H \equiv \frac{\dot a}{a}, \qquad H^2 = \frac{8\pi G}{3}\rho$$

Spatially flat FRW replaces Minkowski space on cosmological scales.

Tensor perturbation

$$g_{ij} = a^2(t)(\delta_{ij} + h_{ij})$$

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j$$

$$\partial_i h_{ij} = 0, \qquad \delta^{ij} h_{ij} = 0$$

The tensor mode is the gravitational-wave sector.

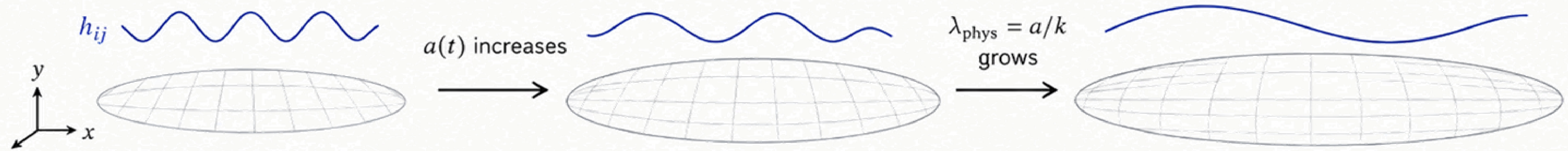
Propagation equation

$$\ddot h_{ij} + 3H\dot h_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 0$$

$$\ddot h_k + 3H\dot h_k + \frac{k^2}{a^2} h_k = 0$$

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0, \qquad \mu_k \equiv a h_k$$

Expansion redshifts physical wavelength and introduces Hubble friction.



Early time (small a)
λ_{phys} small, mode deep inside horizon.
sub-horizon: $h_k \propto 1/a$

Intermediate time
Wavelength grows with expansion; mode approaches the horizon.

Late time (large a)
λ_{phys} large, mode outside horizon.
Amplitude is nearly frozen.

Same tensor perturbation as in flat space, but now the background expansion directly enters the wave dynamics.

Tensor perturbations in FRW

Metric perturbation

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right]$$

$$\partial^i h_{ij} = 0, \qquad h^i{}_i = 0$$

Equation in configuration space

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G a^2 \Pi^{TT}_{ij}$$

In vacuum,

$$\Pi^{TT}_{ij} = 0 \quad \Rightarrow \quad h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 0$$

$\mathcal{H} \equiv a'/a$

Fourier decomposition

$$h_{ij}(\eta, \mathbf{x}) = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^\lambda_{ij}(\hat{\mathbf{k}})\, h_\lambda(\eta, k)\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

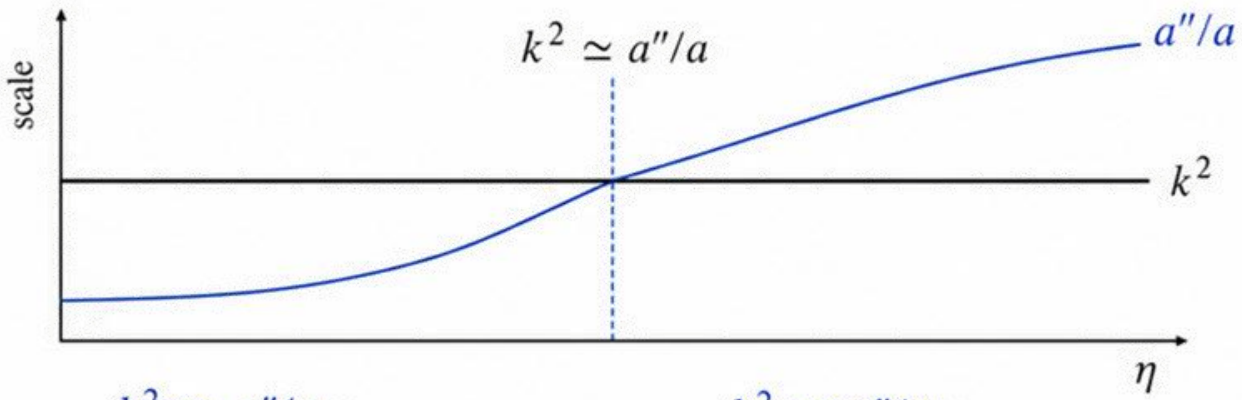
$$h''_\lambda + 2\mathcal{H}h'_\lambda + k^2 h_\lambda = 0$$

Canonical variable and mode equation

$$\mu_\lambda \equiv a\, h_\lambda$$

$$\mu''_\lambda + \left[k^2 - \frac{a''}{a}\right]\mu_\lambda = 0$$

$$\omega_k^2(\eta) = k^2 - \frac{a''}{a}$$



$k^2 \gg a''/a$: oscillatory

$k^2 \ll a''/a$: nearly frozen

The expansion enters through $\mathcal{H} = a'/a$ and through the effective potential a''/a.

Canonical variable for tensor modes

Start in conformal time

$$h_k'' + 2\mathcal{H}h_k' + k^2 h_k = 0$$

$$\mathcal{H} \equiv a'/a, \qquad ' \equiv d/d\eta$$

The damping term is fixed by the expansion.

Rescale the mode

$$\mu_k(\eta) \equiv a(\eta)\, h_k(\eta)$$

The scale factor is absorbed into the field amplitude.

field redefinition →

Canonical equation

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0$$

$$\omega_k^2(\eta) = k^2 - \frac{a''}{a}$$

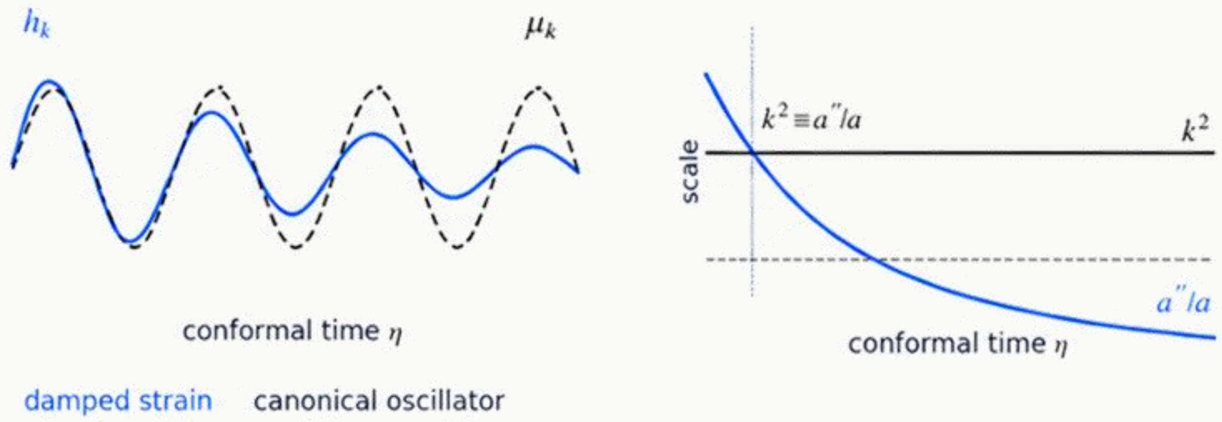
Friction is removed; curvature remains as a potential.

What the rescaling tells us

Relevant comparison

$$k^2 \text{ vs. } a''/a$$

Substitute $h_k = \mu_k/a$ so that the Hubble-friction term cancels.



$k^2 \gg |a''/a|$
oscillatory propagation

$k^2 \lesssim |a''/a|$
background controls evolution

The expansion is encoded in $\mathcal{H} = a'/a$ and a''/a; no particular normalization of a has physical meaning.

Sub-horizon and super-horizon propagation

The horizon scale separates oscillating gravitational waves from frozen tensor perturbations

Starting point

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0$$

$\mu_k = a h_k$

Compare the fixed comoving wavenumber k with the comoving Hubble scale $\mathcal{H}$.

Horizon scale

$$\mathcal{H} \equiv \frac{a'}{a} = aH$$

$$r_H^{\mathrm{com}} = \frac{1}{\mathcal{H}} = \frac{1}{aH}$$

Horizon crossing: $k \simeq \mathcal{H}$

Equivalent physical condition:

$$\frac{k}{a} \simeq H \quad \Leftrightarrow \quad \lambda_{\mathrm{phys}} \simeq H^{-1}$$

Key idea

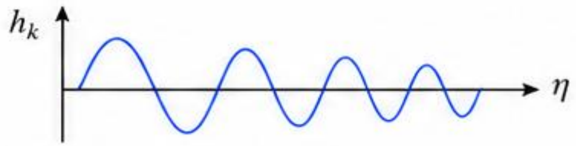
$k^2 \gg \frac{a''}{a} \Rightarrow$ wave-like propagation

$k^2 \ll \frac{a''}{a} \Rightarrow$ frozen tensor amplitude

Two propagation regimes

The horizon separates local wave behavior from frozen super-horizon evolution.

Sub-horizon $(k \gg \mathcal{H})$



Equivalent condition: $\frac{k}{a} \gg H$

k^2 dominates over a''/a

$$\mu_k'' + k^2 \mu_k \simeq 0$$

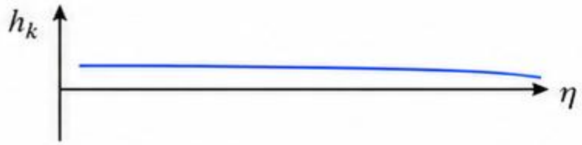
$$\mu_k \simeq A_k e^{-ik\eta} + B_k e^{ik\eta}$$

$$h_k = \frac{\mu_k}{a} \propto a^{-1} e^{\pm ik\eta}$$

Interpretation: genuine oscillating gravitational wave

Amplitude decays as $1/a$ and frequency redshifts as k/a.

Super-horizon $(k \ll \mathcal{H})$



Equivalent condition: $\frac{k}{a} \ll H$

Background term dominates

$$h_k'' + 2\mathcal{H}\, h_k' \simeq 0$$

$$(a^2 h_k')' = 0$$

$$h_k = C_1 + C_2 \int \frac{d\eta'}{a^2(\eta')} \simeq C_1$$

Interpretation: frozen tensor perturbation

There is a constant mode and a decaying mode; the dominant strain is nearly constant.

Inflation: modes exit the horizon as $(aH)^{-1}$ decreases. Radiation/matter eras: modes re-enter as $(aH)^{-1}$ grows. After re-entry, they oscillate again with $h_k \propto a^{-1}$.

Wavelengths stretch with the scale factor

Equations

comoving mode $k = \text{constant}$

comoving wavelength $\lambda_{\rm com} = 2\pi / k$

physical wavelength $\lambda_{\rm phys}(t) = a(t)\,\lambda_{\rm com}$

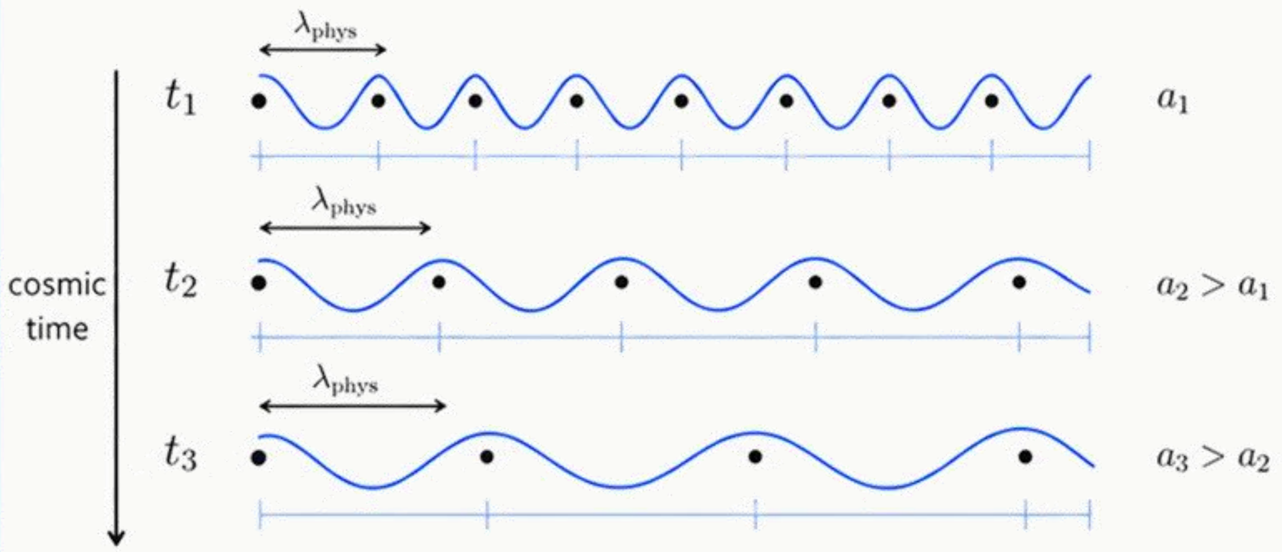
$$\lambda_{\rm phys}(t) = 2\pi\, a(t) / k$$

between emission and observation $\lambda_{\rm obs} / \lambda_{\rm em} = a_0 / a_{\rm em} = 1 + z$

$$f_{\rm obs} = f_{\rm em} / (1 + z)$$

$$M_{\rm obs} = (1 + z)\, M_{\rm source}$$

Geometric picture and consequences



- physical wavelength grows: $\lambda_{\rm phys} \propto a$
- observed frequency decreases
- the waveform is stretched in time
- source masses are observed as redshifted masses

Cosmological expansion leaves the comoving mode label unchanged, but stretches the physical wavelength, lowers the observed frequency, and redshifts the masses inferred from the waveform.

Frequency redshift

Equations

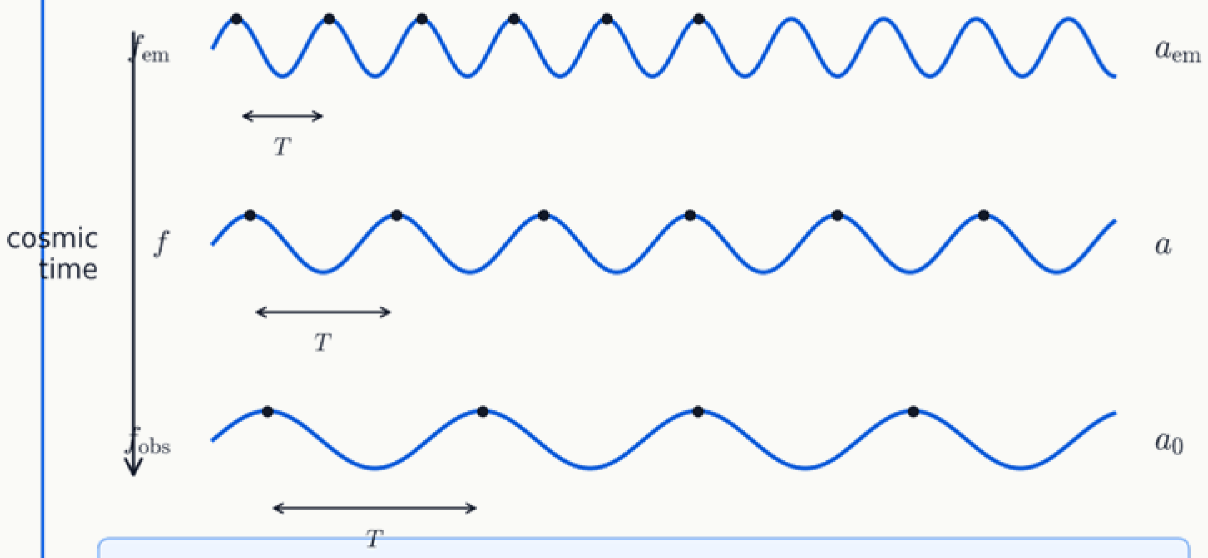
fixed mode	$k = \text{constant}$
physical wavenumber	$k_{\text{phys}}(t) = k/a(t)$
local frequency	$\omega_{\text{phys}}(t) = k/a(t)$

$$\omega_{\text{obs}} = \omega_{\text{em}} \frac{a_{\text{em}}}{a_0} = \frac{\omega_{\text{em}}}{1+z}$$

equivalently	$f_{\text{det}} = f_{\text{src}}/(1+z)$
time intervals	$\Delta t_{\text{det}} = (1+z)\Delta t_{\text{src}}$
interpretation	the same waveform is stretched in time

Geometric picture

Frequency decreases because periods are stretched



$f_{\text{det}} < f_{\text{src}}$ $\Delta t_{\text{det}} > \Delta t_{\text{src}}$

lower pitch, longer duration; polarization content unchanged

Expansion redshifts frequency: $f \propto a^{-1}$. The detector sees a time-stretched, lower-frequency waveform.

Amplitude damping

Scaling laws

sub-horizon mode	$k \gg \mathcal{H}$
strain amplitude	$h_k(\eta) \propto a^{-1}$
physical frequency	$\omega_{\rm phys} = k/a \propto a^{-1}$

$$\rho_{\rm GW} \sim \omega_{\rm phys}^2\, h^2$$

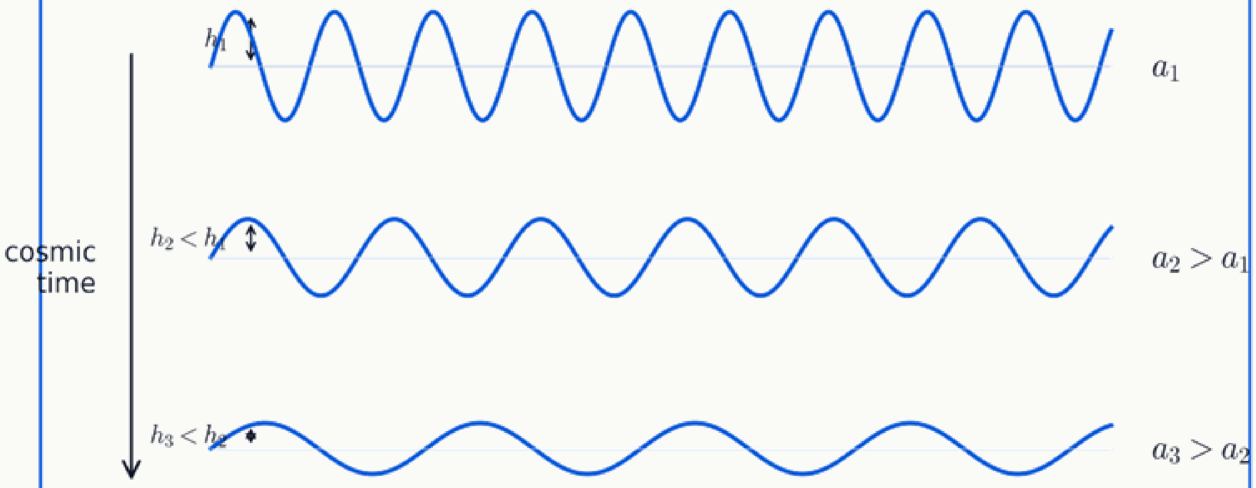
combine $\omega_{\rm phys}^2 \propto a^{-2}, \quad h^2 \propto a^{-2}$

$$\rho_{\rm GW} \propto a^{-4}$$

meaning: freely propagating GWs redshift like radiation

Geometric picture

Amplitude decreases while wavelength increases



$h \propto a^{-1}$ $\omega \propto a^{-1}$ $\rho_{\rm GW} \propto a^{-4}$

two redshift factors for energy density: one from frequency, one from amplitude

For sub-horizon modes, expansion damps the strain as $h \propto a^{-1}$; the gravitational-wave energy density scales as radiation.

Luminosity distance and GW amplitude

Amplitude law

flat spacetime $h \propto 1/r$

FRW spacetime $h_{\rm obs} \propto 1/d_L^{\rm GW}(z)$

$$h_{\rm obs}(t_{\rm obs}) \propto \frac{1}{d_L^{\rm GW}(z)}$$

in standard GR $d_L^{\rm GW}(z) = d_L^{\rm EM}(z)$

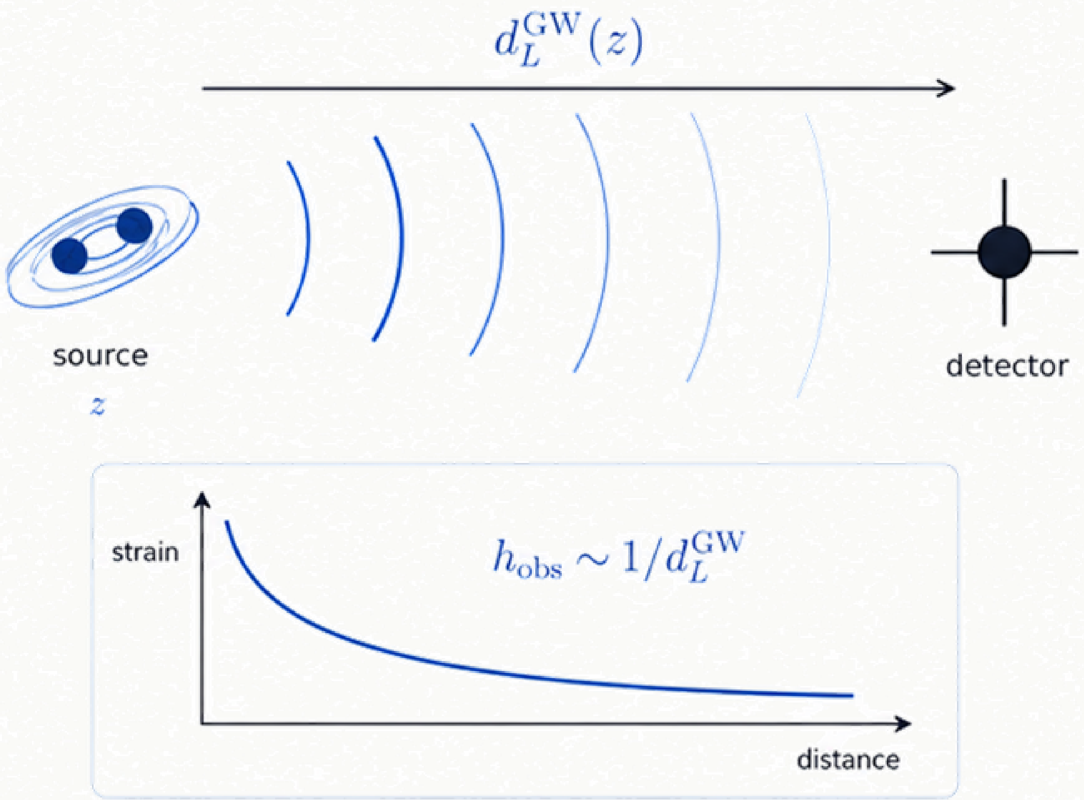
$$d_L^{\rm GW}(z) = (1+z)\int_0^z \frac{c\,dz'}{H(z')}$$

if propagation changes $d_L^{\rm GW} \neq d_L^{\rm EM}$

extra friction changes the amplitude damping

Geometric picture

A standard siren: amplitude gives distance



The observed GW amplitude is calibrated by $d_L^{\rm GW}$. In standard GR it equals the electromagnetic luminosity distance.

Redshifted masses in GW astronomy

Why the chirp mass is redshifted

1 Source-frame chirp law:

$$\frac{df_{\rm src}}{dt_{\rm src}} \propto \mathcal{M}_{\rm src}^{5/3}\, f_{\rm src}^{11/3}$$

2 Cosmological redshift relations:

$$f_{\rm obs} = \frac{f_{\rm src}}{1+z} \qquad dt_{\rm obs} = (1+z)\,dt_{\rm src}$$

3 Therefore:

$$\frac{df_{\rm obs}}{dt_{\rm obs}} = \frac{1}{(1+z)^2}\frac{df_{\rm src}}{dt_{\rm src}}$$

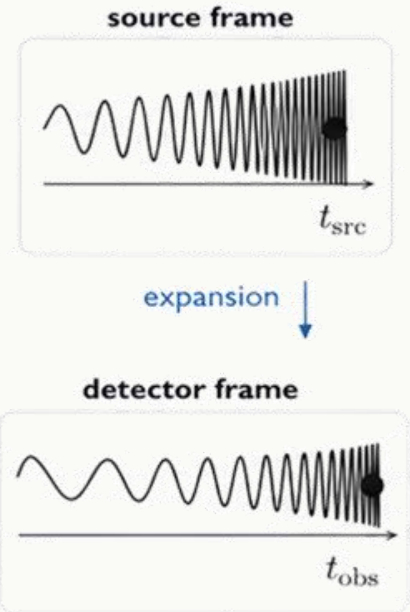
Substitute $f_{\rm src} = (1+z)\, f_{\rm obs}$:

$$\frac{df_{\rm obs}}{dt_{\rm obs}} \propto (1+z)^{5/3}\, \mathcal{M}_{\rm src}^{5/3}\, f_{\rm obs}^{11/3}$$

4

$$\mathcal{M}_z \equiv (1+z)\,\mathcal{M}_{\rm src}$$

The detector measures the redshifted chirp mass, not the source-frame mass.



Cosmic expansion stretches the waveform in time, sot the binary looks slower and therefore heavier.

From the mode equation to $h \propto 1/D_L$

A. Mode equation in conformal time:

$$h_k'' + 2\mathcal{H} h_k' + k^2 h_k = 0, \qquad \mathcal{H} = \frac{a'}{a}$$

B. Canonical field definition:

$$\chi_k(\eta) \equiv a(\eta) h_k(\eta)$$

$$\Rightarrow \quad \chi_k'' + \left(k^2 - \frac{a''}{a}\right)\chi_k = 0$$

C. Sub-horizon / WKB regime:

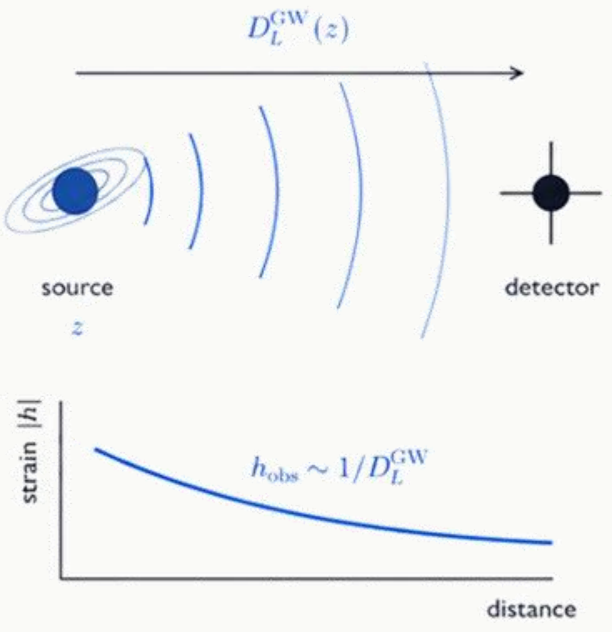
$$k \gg \mathcal{H} \;\Rightarrow\; \chi_k'' + k^2\chi_k \approx 0 \;\Rightarrow\; \chi_k \propto e^{\pm ik\eta}$$

$$\Rightarrow \quad h_k \propto \frac{1}{a}$$

D. Flux and luminosity distance:

$$F_{\rm obs} \sim \frac{c^3}{16\pi G}\omega_{\rm obs}^2 h_{\rm obs}^2 \qquad D_L^2 \equiv \frac{L_{\rm src}}{4\pi F_{\rm obs}}$$

$$\Rightarrow \quad h_{\rm obs} \propto \frac{1}{D_L(z)}$$



In FRW, $D_L(z) = (1+z)\,a_0 r$. The mode equation gives the a^{-1} amplitude damping; the definition of luminosity distance converts this into the observable $1/D_L$ scaling.

A gravitational-wave signal is redshifted in two complementary ways: *its phasing measures the redshifted mass* $\mathcal{M}_z = (1+z)\,\mathcal{M}_{\rm src}$, and its amplitude scales as $1/D_L$, making compact binaries standard sirens.

From propagation to scattering

What changes?

FRW propagation

$$h(\eta,\vec{x}) = \int d^3k\, h_k(\eta) e^{i\vec{k}\cdot\vec{x}}$$

homogeneity selects Fourier modes

Black-hole perturbations

$$\Psi(t,r,\theta,\phi) = \sum_{\ell m} \Psi_{\ell m}(t,r)\, Y_{\ell m}(\theta,\phi)$$

spherical symmetry selects angular multipoles

$$\partial_t^2 \Psi_{\ell m} - \partial_{r_*}^2 \Psi_{\ell m} + V_\ell(r)\Psi_{\ell m} = 0$$

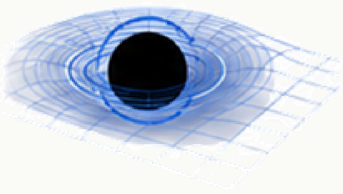
Interpretation $V_\ell(r)$ is a curvature potential.

The radial wave is partly reflected, partly transmitted, and partly absorbed by the horizon.

$$\text{propagation} \longrightarrow \text{scattering}$$

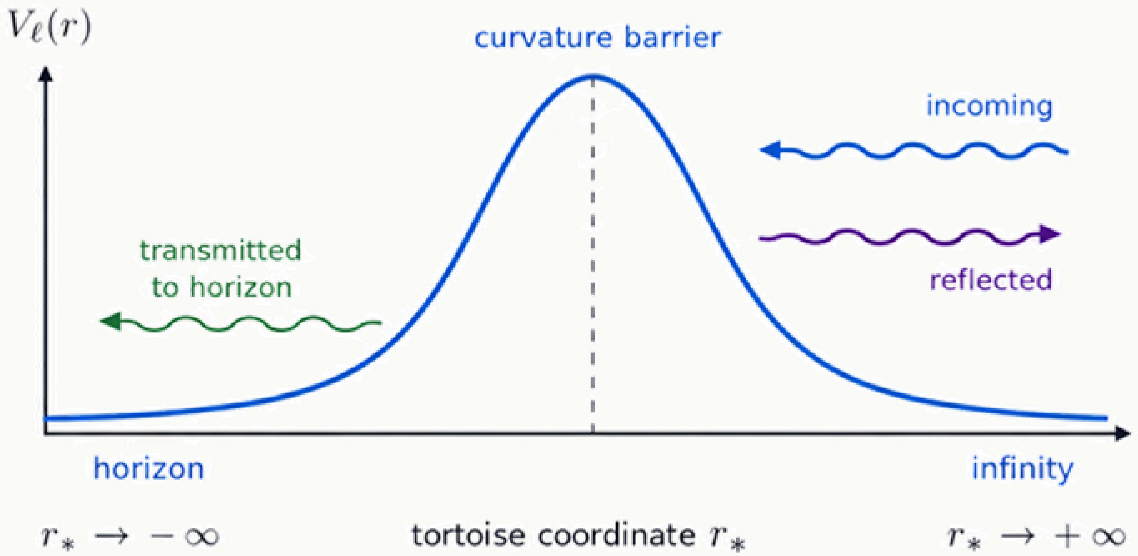
Scattering picture

Effective potential in the tortoise coordinate



$$g_{\mu\nu} = \bar{g}^{\rm BH}_{\mu\nu} + h_{\mu\nu}$$

curvature provides an effective barrier (not a passive medium)



In FRW we tracked redshifted Fourier modes. Near a black hole, curvature turns waves into a scattering problem.

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Perturbations of compact objects

Background and perturbation

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2M}{r}$$

$$g_{\mu\nu} = \bar{g}^{\rm Schw}_{\mu\nu} + h_{\mu\nu}$$

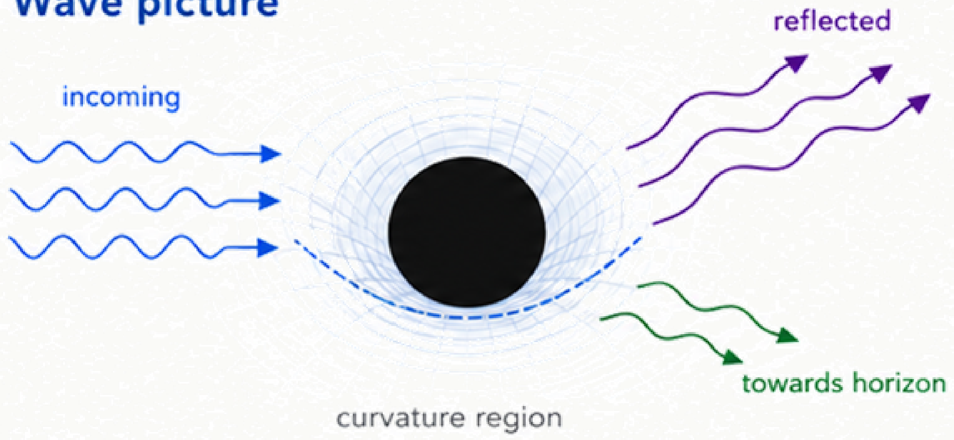
Natural variables

$$e^{i\vec{k}\cdot\vec{x}} \longrightarrow Y_{\ell m}(\theta,\phi)$$

$$h_{\mu\nu} \longrightarrow \{\ell, m, \text{parity}\}$$

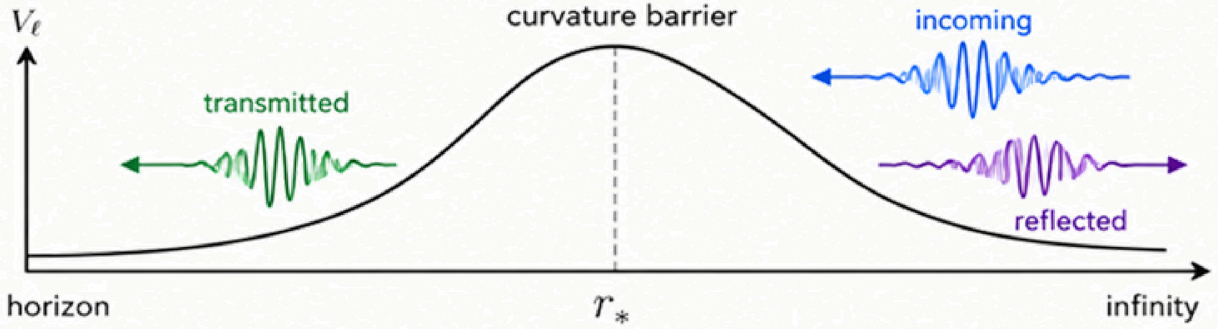
odd / axial sector   even / polar sector

Wave picture



Effective-potential picture

$$\partial_t^2 \Psi_{\ell m} - \partial_{r_*}^2 \Psi_{\ell m} + V_\ell(r)\,\Psi_{\ell m} = 0$$



$h_{\mu\nu}$ metric perturbation → (ℓ, m) and parity angular sectors → $\Psi^{\rm RW/Z}_{\ell m}$ master field → **scattering** open boundaries

The black-hole exterior behaves like an open one-dimensional scattering system.

Normal modes versus quasinormal modes

Curvature converts perturbations into a one-dimensional wave problem with an effective barrier.

Master equation

After spherical-harmonic decomposition, the dynamics is encoded in a master field $\Psi_{\ell m}(t,r)$.

$$\left[-\partial_t^2 + \partial_{r_*}^2 - V_\ell(r)\right]\Psi_{\ell m}(t,r) = S_{\ell m}(t,r)$$

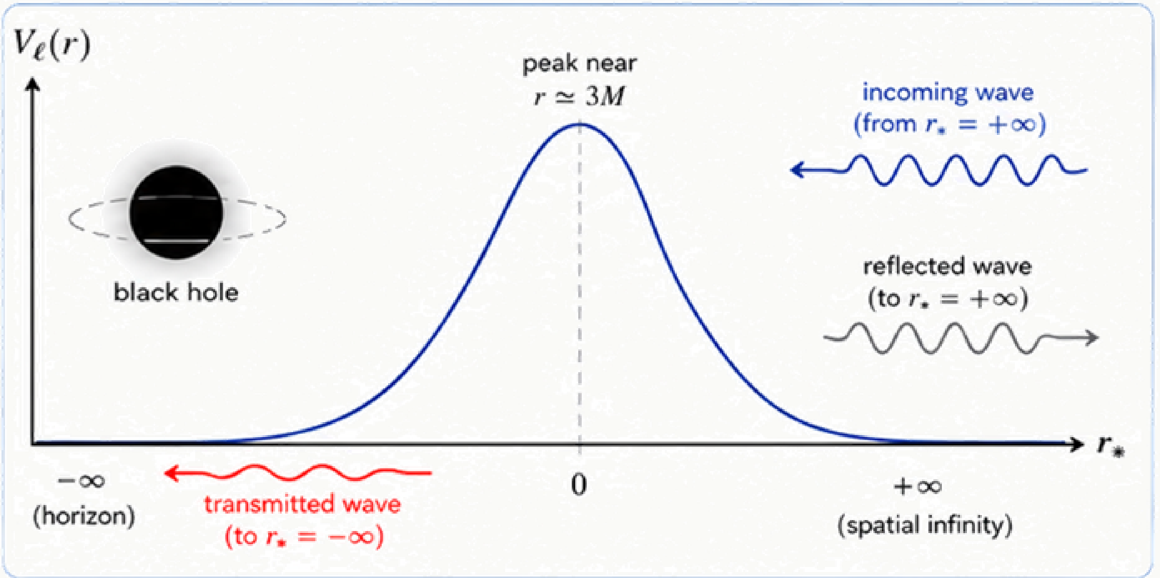
$$r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

Key points

- Separation into spherical harmonics (ℓ, m) reduces the problem to $1+1$ dimensions.
- The tortoise coordinate r_* maps the exterior region $r \in (2M,\infty)$ to $r_* \in (-\infty,\infty)$.
- The effective potential $V_\ell(r)$ encodes spacetime curvature and the centrifugal barrier.

Effective potential barrier

The tortoise coordinate stretches the horizon to the left. The peak of $V_\ell(r)$ lies near $r \simeq 3M$, producing scattering.



Asymptotic regions are flat ($V_\ell \to 0$), so solutions behave as free waves.

Curvature creates an effective potential barrier, so perturbations behave as a 1D scattering problem:
incoming wave → reflection + **transmission** (ringdown).

Quasi Normal Modes

After the source is gone, the black hole rings at its own complex frequencies

Definition

$$\frac{d^2\Psi_\ell}{dr_*^2} + \left[\omega^2 - V_\ell(r)\right]\Psi_\ell = 0$$

A quasinormal mode is an eigenfunction with

horizon	$\Psi_\ell \sim e^{-i\omega r_*}$	ingoing
infinity	$\Psi_\ell \sim e^{+i\omega r_*}$	outgoing

$$\omega_{\ell n} = \omega^{\mathrm{R}}_{\ell n} + i\,\omega^{\mathrm{I}}_{\ell n}$$

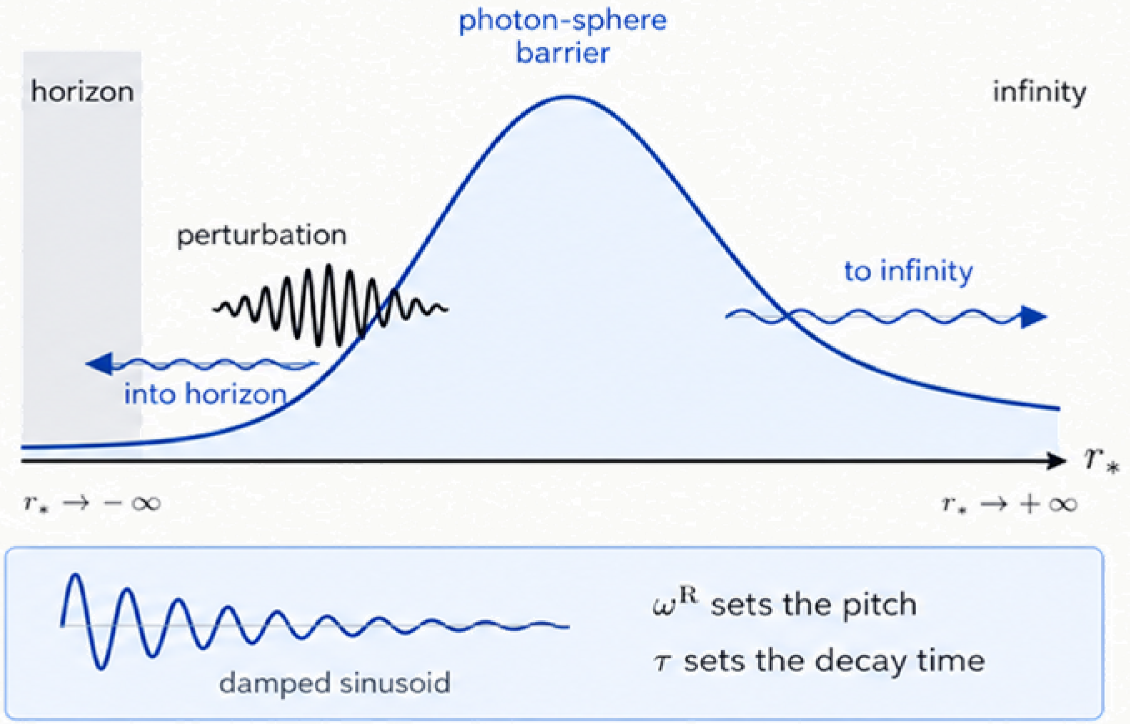
with the convention $e^{-i\omega t} \Rightarrow \omega^{\mathrm{I}}_{\ell n} < 0$

time signal $\Psi(t) \sim A\,e^{-t/\tau_{\ell n}}\cos(\omega^{\mathrm{R}}_{\ell n} t + \phi)$

spectrum fixed by $M,\ a,\ \ell,\ n$ not by the details of the source

Ringing as leakage from a barrier

No incoming waves: energy leaks to the horizon and to infinity



Quasinormal modes are not normal modes of a box. They are resonances of an open spacetime.

Black holes spectroscopy

Kerr spectrum

In GR, a stationary vacuum black hole is Kerr:

$$\omega^{\text{phys}}_{\ell mn} = \frac{c^3}{GM}\,\hat{\omega}_{\ell mn}(\chi)$$

The allowed complex frequencies are fixed by M, χ and by the labels

$$\ell,\ m,\ n \quad (n=0 \text{ fundamental})$$

Ringdown waveform

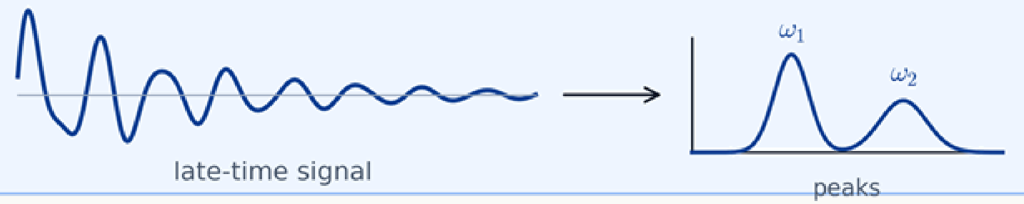
$$h(t) = \sum_{\ell mn} A_{\ell mn} e^{-(t-t_0)/\tau_{\ell mn}} \cos[\omega^R_{\ell mn}(t-t_0) + \phi_{\ell mn}]$$

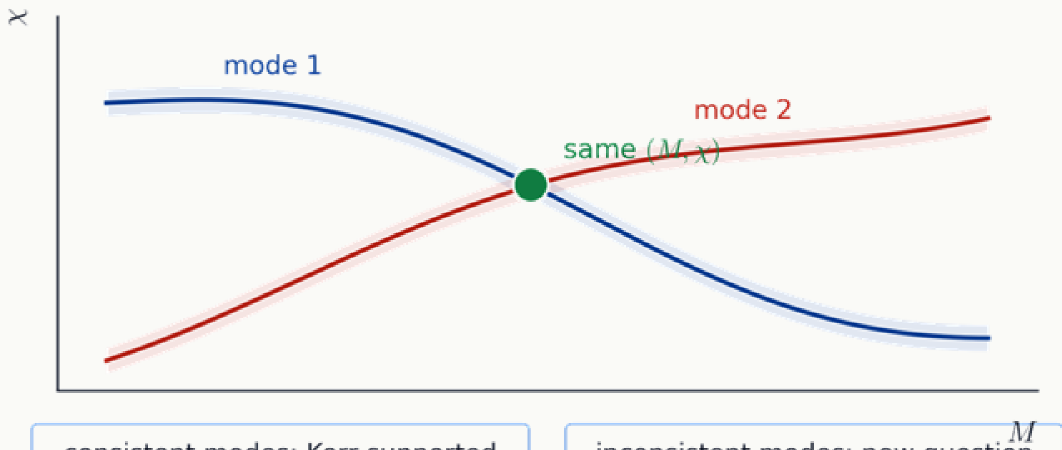
Excitation data	$A_{\ell mn},\ \phi_{\ell mn}$	depend on the merger
Spectral data	$\omega^R_{\ell mn},\ \tau_{\ell mn}$	depend on the remnant

$$\text{one remnant} \implies \text{one Kerr spectrum}$$

Consistency test

Measure several modes and ask whether they agree on the same remnant





consistent modes: Kerr supported

inconsistent modes: new question

Black-hole spectroscopy tests whether all measured ringdown modes fit one Kerr remnant.

Modified gravity roadmap

Front. Astron. Space Sci. 5:44 (2018)

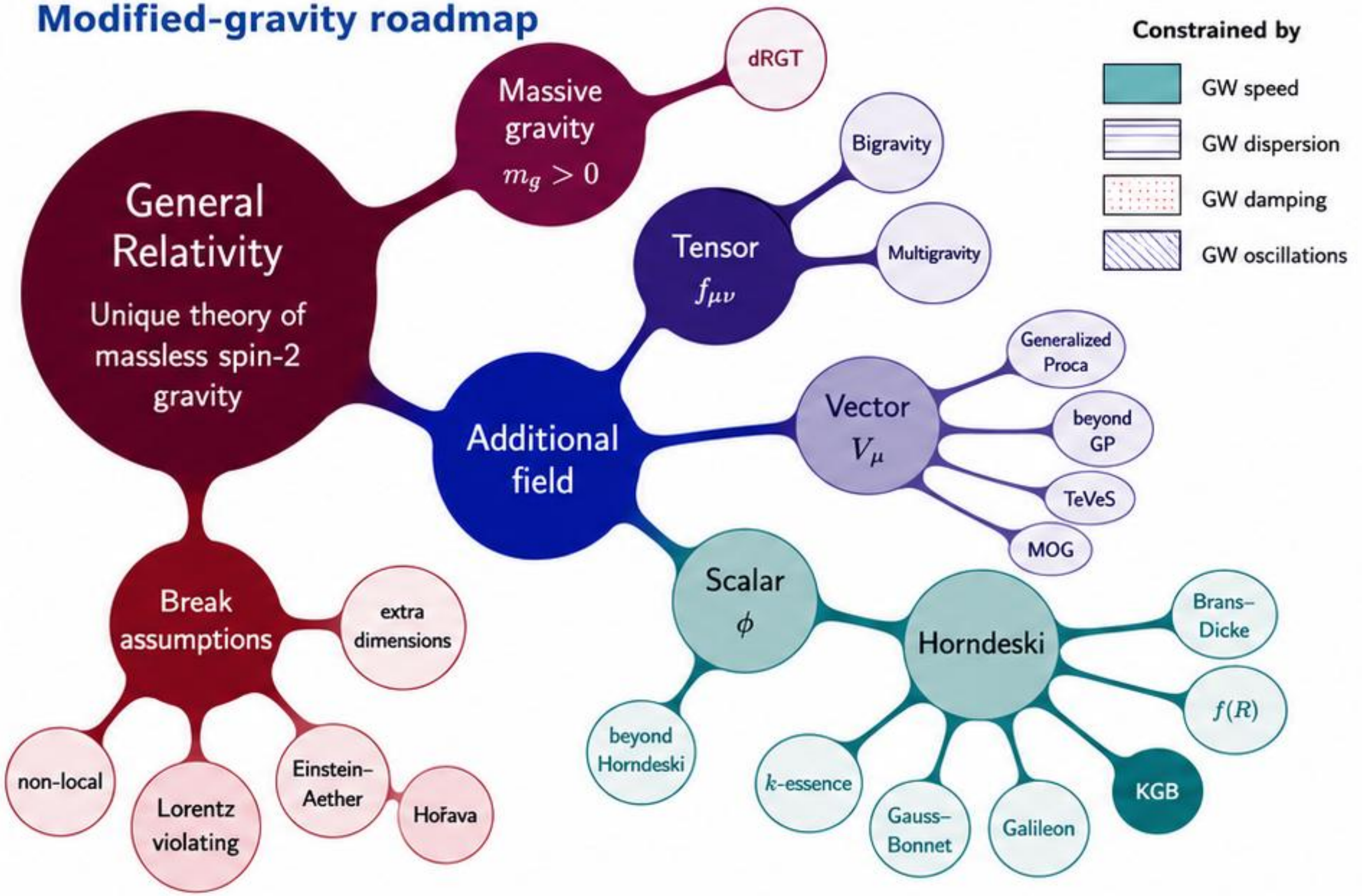
Gravitational waves provide new tests of General Relativity and of possible extensions

Why gravitational waves matter

- Probe **dynamical** and **strong-field** regimes.
- Test signals generated in the **very early Universe**.
- Enable **multimessenger** tests with standard sirens.
  - propagation speed
  - amplitude damping
  - additional polarizations
  - GW oscillations

GW observations constrain how gravity propagates, not only how it generates radiation.

Modified-gravity roadmap



Warning: it is still difficult to place precise constraints on specific alternative theories.

Non standard polarizations